Electoral Competition with Entry under Non-Majoritarian Run-Off Rules[1]

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Abstract

I study electoral competition between established parties under threat of entry by a challenger under *non-majoritarian* run-off elections. In contrast with classical majoritarian run-off elections, I show how non-majoritarian rules can facilitate two-party systems, including those in which the established parties deter the challenger's entry by adopting differentiated platforms. I also show that non-majoritarian run-off rules may facilitate entry deterrence by established parties in settings where a plurality rule cannot. My results provide a striking counterpoint to a conventional wisdom—embodied, most notably, in *Duverger's Hypothesis*—that associates run-off rules with multi-party systems. Finally, they provide a theoretical foundation for patterns of electoral competition observed in countries using non-majoritarian rules that contradict this conventional wisdom.

Keywords: Run-off, Electoral Competition, Entry Deterrence

1. Introduction

In this paper, I examine entry and positioning by established parties facing a threat of entry by a new party (a "challenger") in an important class of *non-majoritarian run-off* elections. In a traditional run-off election, there are at most two rounds of balloting. If a party wins a (strict) majority of the vote in the first round, the contest ends and this party wins the election. If no party wins a majority, the two parties winning the highest and second highest share of the vote compete under a majority rule in a second round. The plurality that a party must win in the first round in order to avoid a second round contest is called the *threshold of exclusion.* The majoritarian threshold, for example, is used for presidential elections in France, Austria and Russia.

Yet a number of countries have adopted substantially different forms of the run-off rule. In Costa Rican presidential elections, only a plurality exceeding 40% is needed for a victory in the first round. In Argentinian presidential elections, a runoff is required unless the winner has either a 45% plurality, or instead a 40% plurality combined with at least a 10% lead over the strongest competitor. In 1999, Nicaragua weakened its run-off plurality threshold of 45% to instead requiring either a 40% plurality, or instead 35% plus a 5% lead over the strongest competitor. In Ecuador the requirement is 40% of the vote and a 10% lead over the strongest competitor. In Mongolian parliamentary elections between 1996 and 2004, the plurality

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requirement was only 25%.[2] Non-majoritarian run-off variants are also found in the United States: in South Dakota primaries for federal and gubernatorial races, a plurality of 35% is required to end the contest; a 40% threshold applies in North Carolina primaries and prior to 2008, Kentucky operated a 40% threshold for gubernatorial primaries.

Majoritarian run-off elections have long been understood to promote multi-party systems, a conviction most famously embodied in *Duverger's Hypothesis* (Duverger (1954)). This association has received both empirical support (eg. Riker (1982), Shugart and Carey (1992), Wright and Riker (1989), Fujiwara (2011), Bordignon, Nannicini and Tabellini (2016)) and theoretical foundations (Haan and Volkerink (2001), Callander (2005*a*), Brusco, Dziubinski and Roy (2012)). Does it extend to non-majoritarian contexts? In Argentina, electoral reform from a plurality rule to its current non-majoritarian run-off rule for electing presidents was made "with the aim of maintaining a two-party system with [Justicialist Party] as the dominant party" (Negretto, 2004, 110). Similarly, Costa Rica switched from a 50% threshold to a 40% threshold in 1936: after the 1948 civil war, the country remained a two-party system for the rest of the twentieth century. In fact, the two leading parties—the PLN and PUSC— jointly won more than 85% of the first-round presidential vote in all but one election over the period 1953 to 1998. McClintock (2007) uncovers similar patterns across other Latin American countries using non-majoritarian run-offs.

I rationalize these observations in a Downsian model of electoral competition with endogenous participation in which there are two or more established parties, and a challenger. I show how non-majoritarian run-off rules may be consistent with—or uniquely predict—two-party systems, in sharp contrast with majoritarian rules. I also characterize properties of multi-party equilibria. Most notably, I show how non-majoritarian rules may allow established parties to coordinate on differentiated platforms that deter the entry of the challenger.

My focus on entry deterrence extends earlier work by Palfrey (1984), Weber (1997), Osborne (2000) and Callander (2005*b*) to a previously unstudied set of electoral institutions. Majoritarian run-off rules are studied in a Downsian framework by Osborne and Slivinski (1996), Haan and Volkerink (2001), Callander (2005*a*), Brusco, Dziubinski and Roy (2012) and Solow (2014). Callander (2005*a*) obtains entry deterrence by two established parties under a classical majoritarian run-off under the assumption that voters always resolve indifference between multiple parties in favor of an established party over a challenger. Otherwise, the challenger always enters. Recent work on strategic voting and information aggregation includes Bouton (2013), Bouton and Gratton (2015), Martinelli (2002) and Morton and Rietz (2006).

2. Model

Preliminaries. The policy space is $\mathbb{R}$. There is a set $N = \{1, ..., n\}$ of established parties, a single challenger c, and a continuum of voters. Voters have continuous, symmetric, single-peaked preferences over policies in $\mathbb{R}$, and their unique ideal policies are distributed over $\mathbb{R}$ according to a continuous and strictly increasing distribution function F, where $r_z \equiv F^{-1}(z)$.

Timing. There are three periods. In period $t = 1$, each established party simultaneously selects an action $a_i \in \mathbb{R} \cup \{out\}$, where an action in $\mathbb{R}$ is interpreted as a policy, and the action *out* is interpreted as withdrawing from the contest. A decision to withdraw can most naturally be interpreted as consolidating support behind another established party. In period $t = 2$, the challenger selects an action $a_c \in \mathbb{R} \cup \{out\}$, where the action

[2] These and other examples appear in Lagerspetz (2016), pp. 61-62, and McClintock (2007).

$a_c = out$ indicates that the challenger does not contest the election. In period $t = 3$, voters cast their ballots. H^t denotes the set of all period-t histories, where h^1 is the null history. A period-2 history is $h^2 = (a_1, ..., a_n)$, a record of the action taken by each established party. A period-3 history is $h^3 = (a_1, ..., a_n, a_c)$, a record of the action taken by each established party and the challenger, a_c. The function $\varrho : \mathbb{R} \times H^t \to \mathbb{N}$ associates with each $x \in \mathbb{R}$ the number of parties located at platform x after history $h^t \in H^t$; $X(h^t) = \{x_1, ..., x_k\}$ denotes the ordered set of platforms such that $x \in X(h^t)$ if and only if $\varrho(x, h^t) \geq 1$. A *policy profile* associated with history h^3 is $\mathcal{X}(h^3) = (X(h^3), \varrho(h^3), a_c)$.

The Election. In period 3, each voter observes the policy profile $\mathcal{X}(h^3)$ and votes for the party that she weakly prefers to all other parties; if she weakly prefers more than one party, she randomizes uniformly over these parties. For any period-3 policy profile (for notational parsimony, I suppress the reference to h^3) $\mathcal{X} = (X, \varrho, a_c)$, the *left semi-constituency* of platform $x_i \in X = \{x_1, ..., x_k\}$ is $F(x_i) - F(.5(x_{i-1} + x_i))$ if $i \geq 2$, or $F(x_i)$ if $i = 1$. The *right semi-constituency* of platform x_i is $F(.5(x_i + x_{i+1})) - F(x_i)$ for $i \leq k - 1$, and $1 - F(x_i)$, if $i = k$. The first-round vote share of a party i that chooses $a_i \in \mathbb{R}$, denoted $v_i(\mathcal{X})$, is equal to the sum of the associated platform's left and right semi-constituencies, divided by the number of parties that locate at that platform. A party that chooses $a_i = out$ receives a vote share of zero.

The electoral rule is summarized by the tuple (p, q), where $p \leq \frac{1}{2}$ and $q \leq \frac{1}{2} - p$. Let $W(\mathcal{X}; p, q)$ denote the set of parties such that:

$$v_i(\mathcal{X}) \begin{cases} > p & \text{if } p = .5 \\ \geq p & \text{if } p < .5 \end{cases}$$

and

$$v_i(\mathcal{X}) - \max_{j \neq i} v_j(\mathcal{X}) \geq q. \tag{1}$$

If $W(\mathcal{X}; p, q) \neq \emptyset$, each party in $W(\mathcal{X}; p, q)$ is equally likely to be selected as the winner. Otherwise, the winner is the party that obtains a majority in the second round in a contest between the two parties that obtained the most votes in the first round. So, for example: Costa-Rica corresponds to $p = .4$, $q = 0$, and Ecuador corresponds to $p = .4$, $q = .1$. Any ties in both rounds are resolved fairly.[3]

Party Preferences, Strategies and Equilibrium. Each party's payoff is its probability of winning; I further assume that a party strictly prefers to enter if it wins with (strictly) positive probability, but strictly prefers to remain out rather than enter the contest and win with probability zero. I study subgame perfect equilibria in pure strategies. I say that $\mathcal{X}$ is an equilibrium profile if there exists a subgame perfect equilibrium that supports the period-3 policy profile $\mathcal{X}(h^3) = \mathcal{X}$.

Discussion. My model shares with Palfrey (1984), Weber (1997), and Callander (2005*a*) an extensive form in which established parties simultaneously make location decisions under the threat of a single challenger. An established party can be interpreted as one of a small number of parties that exist at the inception of a new democratic constitution—in Costa Rica, for example ($p = .4$, $q = 0$) the PLN formed shortly after the 1948 civil war and the country's return to democracy. The challenger is interpreted as a potential new

[3]So, if two or more parties belong to $W(\mathcal{X}; p, q)$ each is equally likely to win in the first round. If a single party wins the highest share of the vote in the first round, but does not belong to $W(\mathcal{X}; p, q)$, the rule randomizes uniformly on the set of parties which win the second highest share of the vote in order to select the party which will compete in the second round against the party which wins the highest vote share in the first round. If two parties win the highest share of the vote in the first round but none belong to $W(\mathcal{X}; p, q)$, each of these parties proceed to a second round with probability one. If more than two parties win the highest share of the vote in the first round but none belong to $W(\mathcal{X}; p, q)$, the rule randomly assigns two of these parties to the second round.

party that is attempting to gain a foothold in an established party system. For example, in Costa Rica, the PUSC formally organized for the first time in 1984, only after the PLN had established itself as a dominant established political party.

My benchmark presentation considers a single challenger: I later show how my results extend when there are two challengers, and I also consider other variants of non-majoritarian run-off rules as well as a plurality rule. Proofs of the main results are in the Appendix, with additional extensions and results in a Supplemental Appendix.

3. Results

Two Established Parties. I start by considering a setting with two established parties facing a single challenger, i.e., a setting with $n = 2$.

Proposition 0. Under a *majoritarian* run-off rule, i.e., $p = .5$, a continuum of equilibria exist. In every equilibrium, both established parties and the challenger win the election with positive probability.

1. An equilibrium exists in which each established party and the challenger locate at $r_{\frac{1}{2}}$, the median voter's most preferred policy.
2. An equilibrium exists in which one established party locates at $x_1 \in (x^*, r_{\frac{1}{2}})$, for some $x^* < r_{\frac{1}{2}}$, the other established party locates at $x_2 > r_{\frac{1}{2}}$ satisfying $x_2 - r_{\frac{1}{2}} = r_{\frac{1}{2}} - x_1$, and the challenger locates at $a_c \in \{x_1, x_2\}$.

In any other equilibrium, one established party locates at $x_1 < r_{\frac{1}{2}}$, the other established party locates at platform x_3 solving $x_3 - r_{\frac{1}{2}} = r_{\frac{1}{2}} - x_1$, and the challenger locates at $x_2 \in (x_1, x_3)$.

The Proposition is consistent with *Duverger's Hypothesis*—that the majority run-off favors multi-party systems—though I will show that this depends critically on the restriction to run-off rules in which the threshold of exclusion is majoritarian. The first class was originally identified by Haan and Volkerink (2001), while the second class was originally identified by Callander (2005*a*). I illustrate each class by way of example, in which voters' ideal policies are uniformly distributed over the unit interval.

Consider the first class, in which both established parties locate at .5—the median voter's most preferred policy—and the challenger also locates at .5. If the challenger were to deviate to any other location, it may win a first-round plurality but cannot obtain a majority. It is therefore defeated in any second round contest. Similarly, if either established party were to locate at another platform, the challenger's best response ensures that the established party loses the election. Consider, instead, the second class. In the uniform example, one established party locates at $x \in [.25, .5)$, and the remaining established party locates at $1 - x$. The challenger locates at either x or $1 - x$. The smallest platform $x = .25$ consistent with this equilibrium—and implicitly defined for general distributions in the proof of the Proposition—is determined by the condition that the challenger cannot profitably locate at a platform on the interior of x and $1 - x$, and come at least second place in the first round. The possibility of a third class of equilibria with three platforms depends sensitively on the distribution of voter preferences—with a uniform distribution, for example, every equilibrium takes the form specified in point 1 or point 2.

I next establish a stark contrast with a non-majoritarian setting.

Proposition 1. Under a *non-majoritarian* run-off rule satisfying $p < .5$ and $0 \leq q \leq \frac{1}{6}$, in every equilibrium, only two parties win the election with positive probability.

1. If $p < .5$ and $q \in [0, .25)$, an equilibrium exists in which one established party and the challenger locate at the median voter's most preferred policy, $r_{\frac{1}{2}}$, and the remaining established party chooses *out*.

2. If $p < .5$ and $q \in (0, .25)$, an equilibrium exists in which one established party locates at $x_1 \in (x^*, r_{\frac{1}{2}})$, for some $x^* \in [r_{\frac{1}{2}-q}, r_{\frac{1}{2}})$, the other established party locates at $x_2 > r_{\frac{1}{2}}$ satisfying $x_2 - r_{\frac{1}{2}} = r_{\frac{1}{2}} - x_1$, and after the established parties choose these locations, the challenger chooses $a_c = out$.

If $p < .5$ and $q \leq \frac{1}{6}$, there are no other equilibria.

Thus, there is a discontinuity in the properties of run-off equilibria at the majoritarian threshold. Consider, first, a rule with a plurality threshold $p < .5$, but no margin requirement, i.e., $q = 0$. After a history in which a single established party locates at the median voter's most preferred policy, the challenger can do no better than choose the same location. Moreover, if the other established party were to remain in the race by locating at a platform, it would merely create an opportunity for the challenger to enter the contest and win in the first round.

A positive margin requirement, i.e., $q > 0$, creates the possibility for both established parties to deter the challenger's entry. To illustrate the result, consider Ecuador's run-off rule that requires no second round if and only if a party wins 40% in the first round, and leads the second-highest scoring party by at least ten percentage points, i.e., $p = .4$ and $q = .1$. There is an equilibrium in which, on the equilibrium path, one established party locates at $x \in (.5 - q, .5)$, the other established party locates at $1 - x$, i.e., the symmetric location on the opposite side of the median voter's ideal point, and the challenger stays out.

It is easy to show that, after this action profile by the established parties, the challenger's unique best response is to stay out: locating at $a_c \leq x$, or $a_c \geq 1 - x$ either hands the election to one of the established parties in the first round, or ensures the challenger's defeat in a second round. Similarly, $a_c \in (x, 1 - x)$, leads to the challenger's elimination in the first round, since she wins only the third highest share of the vote.

Could one of the two established parties located at either $x \in (.5 - q, .5)$ or $1 - x$ do better, given the other's initial choice of location? Consider a deviation by the established party that locates at x to some other policy, $x' \in [0, 1]$. It is easily verified that if $x' < x$ or $x' > .5$, for any best response of the challenger, the established party located at x' is no better off. Consider, however, a location $x' \in (x, .5]$. I specify that, after this history, the challenger locates at any platform $a_c \in (x, x')$. In the election, the established party located at x' comes third and is therefore eliminated. Either the established party located at $1 - x$ or the challenger located at a_c wins a plurality in the first round; but even if the established party located at $1 - x$ wins a plurality, its margin of victory over the challenger is strictly less than $.5 - x = .5 - (.5 - q) = q$. This triggers a second round contest between the challenger, located at a_c, and the established party, located at $1 - x$. In that second round, the challenger wins, since its platform $a_c \in (x, x')$ is closest to the median voter's most preferred policy. Thus, a positive margin requirement $q > 0$ disciplines each established party from locating closer to the median voter's most preferred policy, given the other established party's location decision.

Many Established Parties. I first show that the equilibria identified in points 1. and 2. of Proposition 1 extend to a setting with *any* number $n \geq 2$ of established parties.

Corollary 1. Points 1. and 2. of Proposition 1 extend to a setting with any number $n \geq 2$ of established parties:

1. If $p < .5$ and $q \in [0, .25)$, an equilibrium exists in which one established party and the challenger locate at the median voter's most preferred policy, $r_{\frac{1}{2}}$, and the remaining established parties choose *out*.
2. If $p < .5$ and $q \in (0, .25)$, an equilibrium exists in which one established party locates at $x_1 = (x^*, r_{\frac{1}{2}})$, for some $x^* \in [r_{\frac{1}{2}-q}, r_{\frac{1}{2}})$, another established party locates at $x_2 > r_{\frac{1}{2}}$ satisfying $x_2 - r_{\frac{1}{2}} = r_{\frac{1}{2}} - x_1$, the remaining established parties choose *out*, and after the established parties choose these actions, the challenger chooses *out*.

The first point of Proposition 0 also extends to any number $n \geq 2$ of established parties: under a majoritarian run-off, an equilibrium exists in which each of the $n \geq 2$ established parties and the challenger locate at the median voter's most preferred policy. Thus, the sharp contrast between existence of an equilibrium in which all the established parties and the challenger locate at the median voter's most preferred policy under a majoritarian rule, and only two parties locate at that policy under a non-majoritarian rule, extends. So, too, does the prospect for early entrants to implicitly collude on deterring the entry of a subsequent prospective challenger, if there is a positive margin requirement $q > 0$.

With many established parties, however, it is no longer assured that in every equilibrium, only two parties win with positive probability—in fact, the set of equilibria significantly expands. I focus, next, on characterizing and illustrating properties of equilibria in which more than two parties—including, possibly, the challenger—win with positive probability.

Lemma 1. If $\mathcal{X} = (X, \varrho, a_c)$ is an equilibrium profile, where $X = \{x_1, ..., x_k\}$ is the set of ordered platforms, either (1) all parties that offer platforms tie for first place in the first round, or (2) in the first round, a single established party located at either $x_1 < r_{\frac{1}{2}}$ or $x_k > r_{\frac{1}{2}}$ wins a strict plurality, all remaining parties that offer platforms tie for second place, and the extreme platforms are symmetrically located around the median voter's most preferred policy, i.e., $r_{\frac{1}{2}} - x_1 = x_k - r_{\frac{1}{2}}$.

If a strict plurality-winner were located at an interior platform $x_2, ..., x_{k-1}$ and progressed to the second round with probability one, at least one party positioned strictly further from the median must lose in either a first or second round; this is inconsistent with that party's entry since it would then prefer not to enter the contest. And, if a party located an extreme platform wins a strict plurality, the extreme policies must be symmetrically located around the median voter's most preferred policy in order to ensure that every party located at an extreme policy wins with positive probability.

In Supplemental Appendix A, I provide platform formulae and conditions on the first-round vote shares for equilibria involving *any* number of platforms for *any* number $n \geq 2$ of established parties, where they are pinned down. I illustrate the conditions by way of two examples, for each class of equilibria identified in the previous Lemma. In both examples, I consider a setting with three established parties and a single challenger, and voter ideal policies that are uniformly distributed on the unit interval.

Example 1: Multi-party equilibria in which all parties tie in first round. I first consider an example of equilibria conforming to the first class of Lemma 1, in which the challenger offers a platform, and in which there are two platforms, symmetrically located around the median voter's most preferred policy. One established party locates at a platform $x_1 < .5$, two established parties locate at $x_2 = 1 - x_1$, i.e., symmetrically around the median voter's most preferred policy, and the challenger subsequently locates at x_1.

First, the platforms must be positioned sufficiently far apart that the challenger does not prefer to deviate to a platform in $[0, x_1)$ or in $(x_2, 1]$. To be profitable, such a deviation *must* guarantee the challenger a first-round victory, since it is located strictly further from the median voter's most preferred policy than any other party. Thus, a deviation is not profitable if $x_1 \leq \max\{p, .25 + q\}$.

Second, the platforms must be positioned sufficiently close together that the challenger does not prefer to deviation to a location in (x_1, x_2). The challenger prefers any such location only under two conditions. First, it comes *at least* second place, i.e. its vote share after locating at $a_c \in (x_1, x_2)$, is greater than the vote share of the two established parties located at x_2. Second, the location $a_c \in (x_1, x_2)$ must not allow the established party located at x_1 to win in the first round by surpassing the threshold of exclusion. We obtain the condition:

$$\max\left\{\frac{1}{3}\left(p + \frac{1}{2}\right), \frac{1+q}{4}\right\} \leq x_1 \leq \max\left\{p, \frac{1}{4} + q\right\}. \tag{2}$$

Notice that as the threshold falls (either p or q), *both* the smallest *and* greatest platform differentiation consistent with equilibrium increases.

Example 2: Multi-party equilibria in which a single party wins a plurality in the first round. Next, I illustrate equilibria of the second class elaborated in Lemma 1, by considering conditions for an equilibrium with three distinct policies in which the challenger enters the contest and a single party wins a strict plurality in the first round of the election. I continue to assume a uniform distribution of voter ideal policies on the unit interval, and suppose that there are $n = 3$ established parties. For ease of presentation, I also specialize the set of run-off rules to those satisfying $.25 < p < .5$ and $q = 0$.

By Lemma 1, an equilibrium profile with three distinct platforms in which a single party wins a strict plurality must satisfy the following properties: (1) an established party located at extreme platform x_1 or x_3 wins the strict plurality, (2) the extreme platforms must be symmetric around the median, i.e., $x_3 = 1 - x_1$, and (3) any party that does not win a strict plurality ties for second place. I construct an equilibrium profile in which an established party located at the platform x_1 wins a first-round vote share $\alpha \in (.25, p)$, so that each of the two other established parties and the challenger wins a first-round vote share $\frac{1-\alpha}{3}$. To support this profile, I specify a strategy profile in which (i) three established parties each simultaneously locate at platforms x_1, x_2 and $x_3 = 1 - x_1$, and (ii) after this history, the challenger locates at x_3. Letting $b_i(\mathcal{X}) = .5(x_i + x_{i+1})$ denote the location of a voter that is indifferent between two adjacent platforms x_i and x_{i+1}, for $i \in \{1, 2\}$, we obtain:

$$b_1(\mathcal{X}) = \alpha, \quad b_2(\mathcal{X}) = 1 - \frac{2}{3}(1 - \alpha), \tag{3}$$

and combining with the requirement $x_3 = 1 - x_1$, we obtain the platform locations, as a function of $\alpha \in (.25, p)$:

$$x_1 = \frac{1 + 2\alpha}{6}, \quad x_2 = \frac{5\alpha}{3} - \frac{1}{6}, \quad x_3 = \frac{5 - 2\alpha}{6}. \tag{4}$$

After the established parties choose these locations, the challenger's action $a_c = x_3$ is easily verified to be a best response. But could any of the established parties improve their prospects of winning? If the run-off rule is *majoritarian*, i.e., $p = .5$, the answer is *no*: it is straightforward to show that these actions by each of the established parties and the challenger can be supported as an equilibrium.

What about a *non-majoritarian* rule? Absent the subsequent entry of the challenger, the established party located at x_3 obtains a vote share $\frac{2}{3}(1 - \alpha)$. If $p < \frac{2}{3}(1 - \alpha)$, the established party that is supposed to locate at x_3 could instead relocate slightly to the right, continuing to win strictly more than p vote share and ensuring that the challenger subsequently prefers to stay out, thereby winning the election in the first round. Thus, a necessary condition for this to be an equilibrium profile is $p \geq \frac{2}{3}(1 - \alpha)$, or

$$1 - 1.5p \leq \alpha < p, \tag{5}$$

i.e., $p > .4$. More generally, this profile *could* be supported under Nicaragua's rule $p = .45$ prior to that country's 1999 electoral reform. But, it *could not* be supported in South Dakota ($p = .35$), North Carolina and in Costa Rica ($p = .4$).

More Challengers.[4] Do my results on the prospects for two-party systems under non-majoritarian run-offs hinge critically on the presence of only a single challenger? The answer is *no*. Consider a modification of the game form in which there are four periods. In period 1, as before, each established party makes its location or exit choice. In period 2, the player function assigns one of *two* challengers to choose an action $a \in \mathbb{R} \cup \{out\}$. In period 3, the player function assigns a second challenger to choose an action $a \in \mathbb{R} \cup \{out\}$. In the (terminal) period 4, voters cast their ballots, as in the benchmark setting.[5] I first extend Proposition 1: the proof is contained in Supplemental Appendix C.

Proposition 2. Suppose there are two established parties, and two challengers. If $p < .5$ and $q \in (0, .25)$, there exists $x^* \in (r_{\frac{1}{2}-q}, r_{\frac{1}{2}})$ such that the following is an equilibrium profile: one established party is located at $x_1 = (x^*, r_{\frac{1}{2}})$, the other established party is located at $x_2 > r_{\frac{1}{2}}$ satisfying $r_{\frac{1}{2}} - x_1 = x_2 - r_{\frac{1}{2}}$, and both challengers stay out; moreover, every equilibrium profile has the property that there are at least two distinct platforms.

The Proposition confirms that, with a positive margin requirement $q \in (0, .25)$, the entry deterrence equilibrium characterized in point 2 of Proposition 1 extends to a setting with two challengers. By contrast, point 1 of Proposition 1 does not extend to a setting with a positive margin requirement $q > 0$.

However, if $p < .5$ and $q = 0$, point 1 of Proposition 1 *may* extend to a setting with two challengers. That is, an equilibrium two-party system in which one of the two challengers and one established party locate at the median voter's most preferred policy may exist, depending on the plurality threshold, p. I illustrate by way of an example, and provide additional results in Supplemental Appendix C.

Example 3: If the distribution of voter ideal policies is uniform on $[0, 1]$, there are $n \geq 2$ established parties and two challengers: if and only if $p \leq .375$ and $q = 0$, an equilibrium exists in which a single established party and a single challenger locate at $r_{\frac{1}{2}}$, the median voter's most preferred policy, and the other parties stay out of the race.

To understand the significance of this threshold, consider a setting with two established parties, and a strategy profile in which one established party locates at .5, and the other remains out. After this history, it is straightforward to enforce a decision by the first challenger to remain out, and the second challenger to share the location of the established party at .5. Can an established party that is supposed to quit instead profitably locate at some platform $x \in [0, 1]$, given that one other established party locates at the platform .5? I consider $x \leq .5$, since $x > .5$ follows a symmetric argument.

Consider, in particular, $x \in [.25, .5)$; then the first challenger may locate at y solving $1 - y = .5(x + .5)$, i.e., $y = .75 - .5x$. If the second challenger subsequently stays out, the first challenger wins the election in the first round with a vote share $1 - .5(y + .5) = .375 + .25x > p$. If the second challenger were subsequently to locate at a position $z \geq y$, it wins a vote share strictly less than the party located at x: by $p \leq .375$, the party located at x therefore wins a first round plurality $.5(x + .5) \geq .375$, and thus surpasses threshold of

[4] I am grateful to an anonymous referee for encouraging me to explore this setting.

[5] One might consider an equally natural modeling assumption to be that each challenger must simultaneously make its decision in period 2. However, under this alternative assumption, there are action profiles for the established parties such that in the subgame where each challenger makes its choice, no (pure strategy) Nash equilibrium exists.

exclusion. If, instead, the second challenger were to locate at $z \leq x$, the first challenger located at y wins in the first round. Other locations for the second challenger can easily be ruled out. This implies that, after the entry of the first challenger, the second challenger's best response is to stay out of the contest. Since the second challenger stays out, the first challenger therefore wins in the first round, and the established party's deviation to $x \in [.25, .5)$ is not profitable.

Had the run-off rule instead required a plurality $p > .375$, the established party that is supposed to stay out would instead prefer to locate at .25: the (strict) best response of the first challenger after this history is to locate at .75. After this action, the second challenger strictly prefers to stay out, and the parties located at .25 and .75 each win with probability one half.

No Challengers. The variant of the model with no challengers—i.e., in which all parties compete in a single round—is comprehensively explored in Brusco, Dziubinski and Roy (2012) in the context of a majoritarian rule.[6] I summarize the main difference with my setting. Recall Lemma 1, establishing that in every equilibrium either (1) all parties tie for first place, or (2) a single party wins a first-round plurality, and all remaining parties tie for second place. In a setting where all established parties move simultaneously and face no threat of subsequent entry, there is no equilibrium corresponding to the second class. The reason is that (I earlier showed) a strict plurality-winning party must be located at an extreme platform: absent a threat of subsequent entry, however, this party could re-locate slightly closer to the median voter's most preferred policy, and strictly increase its chances of winning. Thus, in every equilibrium all competing parties tie for first place. In a setting with no challengers, equilibria in pure strategies may also fail to exist. For example, with three parties, there is no pure strategy equilibrium whenever $p < .5$ and $q = 0$. Depending on the run-off rule, existence of a pure strategy equilibrium may still fail with four or more parties. This relates to results by Osborne (1993) on the generic non-existence of pure strategy equilibria in a normal-form setting with three or more parties competing under a plurality rule.

Comparison with a Plurality Rule. I consider the plurality rule with n established parties and a single challenger. Though a full study is beyond the scope of the present paper, I characterize two equilibria which make for an interesting comparison with my main results. I first characterize an equilibrium that corresponds to *Duverger's Law*, and also bears a close relationship to the first class of equilibria characterized in Proposition 1 for a non-majoritarian run-off rule; its proof is a straightforward extension.

Proposition 3. Under a plurality rule, for any number of established parties and a single challenger, an equilibrium exists in which one established party locates at the median voter's most preferred policy, the challenger shares this location and all other established parties stay out. If there are only two established parties, this equilibrium is unique.

Thus, in the case with two established parties, the plurality rule generates two-party systems, just as a non-majoritarian runoff. However, *unlike* non-majoritarian run-off rules with positive margin requirements, i.e., satisfying $q > 0$, the plurality rule cannot support a two-party system in which both of these established parties survive. A striking example of this phenomenon is the demise of the Liberals in the United Kingdom and the rise of Labour as the main opposition to the Conservatives, in the first part of the twentieth century. Thus, a non-majoritarian run-off may be even better-suited to preserving a two-party system of established parties than a plurality rule.

[6] The authors discuss the extension of some of their results to a non-majoritarian setting, in the setting with no challengers.

Proposition 3 shows that with two established parties, a two-party system must arise under a plurality rule. One might suppose that this result should extend to larger number of established parties. For example, if a country were to switch to a plurality rule having inherited a party system with more than two established parties, we might suppose that the plurality rule would force the established parties to consolidate. I show that this is not the case. Suppose that there are three established parties. In contradiction of *Duverger's Law* (but not with reality), an equilibrium may exist in which all three established parties offer platforms, and the challenger also enters the contest. I construct an example, and in Supplemental Appendix D generalize the result to other families of distribution functions.

Example 4: Suppose the distribution of voter preferences is uniform on $[0, 1]$. If there are three established parties and a single challenger, the following is an equilibrium profile: one established party locates at $x_1 = .25$, two established parties locate at $x_2 = .75$, and the challenger locates at $x_1 = .25$.

The requirement that the platforms be $x_1 = .25$ and $x_2 = .75$ can be compared with the equilibrium conditions for the corresponding four-party equilibrium under a non-majoritarian run-off in *Example 1*: see expression (2). While both electoral rules support four-party systems based around one 'leftist' and one 'rightist' platform, the non-majoritarian variant admits less polarized platform locations.

Moving Thresholds.[7] In some countries, the run-off threshold specifies more than one way in which a second round contest is avoided. In Argentinian presidential elections, a party wins in the first round if *either* (1) that party wins a plurality of at least 45%, *or* (2) that party wins a plurality of at least 40% and its vote share lead over the next highest-scoring party is at least 10%. In Nicaraguan presidential elections, since 1999, a party wins in the first if *either* (1) that party wins a plurality of at least 40%, *or* (2) that party wins a plurality of at least 35% and its vote share lead over the next highest-scoring party is at least 5%.

In Supplemental Appendix E, I formally define a run-off rule with a moving threshold. Two-party equilibria always exist under moving thresholds, but they always feature the entry of the challenger and a single established party. Equilibria with more than two parties may also exist.

More than Two Parties in the Second Round.[8] At present, all of the world's non-majoritarian run-off rules specify that if a second round contest takes place, only the two highest-scoring parties from the first round may participate. More generally, however, the rule might specify that if no party surpasses the threshold of exclusion, more than two parties may proceed to the second round. For example, French legislative elections use a majoritarian run-off rule, but allow any candidate with *at least* 12.5% share of the registered electorate in the first-round vote to proceed to a second round; in Hungary, the top three candidates from the first round, plus any candidate whose first-round vote share is above 15%, proceed to a second round.

To illustrate how my results on two-party systems extend to this class of rules, consider a non-majoritarian run-off rule with $p = .35$ and $q = .15$, two established parties and a single challenger. Under a classical non-majoritarian run-off, in which only two parties may contest a second round, there is an equilibrium in which one established party locates at $x_1 = .37$, the other established party locates at $x_2 = .63$, and the challenger stays out, consistent with Proposition 1. Suppose, instead, any party that wins at least a first-round vote share $\theta = .125$ is eligible to participate in the second round if and only if no party surpasses the first round threshold of exclusion. Then, this profile cannot be supported in an equilibrium: for example, the

[7]This nomenclature was first proposed in Bouton (2013).

[8]I am grateful to an anonymous referee for encouraging me to explore this setting.

established party that is supposed to locate at .37 strictly prefers to locate at .38. The reason is that, after a history in which one established party is located at .38 and the other is located at .63, the challenger cannot profitably enter the race. For example, if the challenger locates at $a_c < .38$, it cannot stop the party located at .38 from participating in any second round, since the latter wins a vote share of in excess of $.5(.38 + .63) - .38 = .125 = \theta$. And in such a second round, the challenger located at $a_c < .38$ and the party located at .38 both lose to the remaining party located at .63 by dividing the support of their own constituency. By deterring the challenger's entry after this deviation, the party located at .38 wins the election in the second round.

However, there exist a continuum of equilibria in which one established party chooses $x_1 \in (.5 - \theta, .5)$, the other established party locates at $1 - x_1$, and the challenger stays out. Such a location ensures that if the party that is supposed to locate at x_1 moved slightly towards the median voter's most preferred policy, the challenger could position itself to ensure that the deviating party wins strictly less than θ vote share, and that the remaining established party does not win outright in the first round (since $\theta < q$, in this example).

Conclusions. My theoretical findings are consistent with empirical evidence; more generally, they constitute the first dedicated theoretical analysis of party positioning under this important class of electoral rules. There are at least two promising directions for future research. First, it would be natural to introduce aggregate uncertainty about the preferences of voters in the setting with strategic entry and location choices, as proposed by Solow (2014) in the case of a majoritarian run-off rule. Second, the present model features a continuum of voters: a drawback of this approach is that strategic considerations by voters are ruled out by design. In an important contribution, Bouton (2013) studies strategic voting for plurality thresholds $p \in [\frac{1}{3}, \frac{1}{2}]$ in a setting with three fixed alternatives (candidates). Combining strategic voting with strategic candidacy would be a natural next step.

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4. Appendix: Proofs of Results

I start with a Lemma establishing the existence of a best response for the challenger in the subgame beginning after any period-2 history. **Lemma 2**. In the setting with $n = 2$ established parties, after any history $h^2 \in H^2$, the challenger has a best response. **Proof**. Recall that in period 3, every voter casts her ballot sincerely, and randomizes uniformly over the set of parties between which she is indifferent. Let $\pi(a_c, h^2)$

denote the probability that the challenger wins the election if it chooses $a_c \in \mathbb{R}$ after history h^2, and $\Pi(h^2) = \{\pi \in [0,1] : \pi(a_c, h^2) = \pi$ for some $a_c \in \mathbb{R}\}$. It is easily verified that for any $h^2 \in H^2$, $\Pi(h^2)$ is nonempty and finite, and therefore contains a largest element $\hat{\pi}(h^2)$. Letting $BR(h^2)$ denote the challenger's set of best responses, we conclude $BR(h^2) = \{a_c \in \mathbb{R} : \pi(a_c, h^2) = \hat{\pi}(h^2)\}$ if $\hat{\pi}(h^2) > 0$, and $BR(h^2) = \{out\}$ if $\hat{\pi}(h^2) = 0$. $\square$

Proof of Proposition 0. In the following arguments, recall that $\varrho(x, h^t)$ denotes the number of parties located at platform $x \in X(h^t)$ after a history h^t. Where no explicit reference is made to a history, i.e., $\varrho(x)$, I refer to the total number of parties located at platform x after a period-3 history, i.e., after each of the established parties and the challenger have made entry-location choices. I first show that if $p = .5$, an equilibrium profile $\mathcal{X} = (X, \varrho, a_c)$ satisfies $\sum_{x \in X} \varrho(x) = 3$. *Claim 1.* There is no equilibrium profile satisfying $\sum_{x \in X} \varrho(x) = 1$. *Proof.* If, on the equilibrium path, $a_c = out$, a deviation to $a_c = r_{\frac{1}{2}}$ is profitable. If, instead, $a_c = x_1$ on the equilibrium path, i.e., $\sum_{x \in X(h^2)} \varrho(x, h^2) = 0$, let one established party instead locate at $r_{\frac{1}{2}}$. After this deviation, the challenger's unique best response is $a_c = r_{\frac{1}{2}}$, and the deviation is profitable. $\square$ *Claim 2.* There is no equilibrium profile satisfying $\sum_{x \in X} \varrho(x) = 2$. *Proof.* If $X(h^2) = \{x_1\}$, $x_1 \neq r_{\frac{1}{2}}$, $a_c = r_{\frac{1}{2}}$ is a profitable deviation. If $x_1 = r_{\frac{1}{2}}$ and $a_c = out$, $a_c = r_{\frac{1}{2}}$ is a profitable deviation. If $x_1 = r_{\frac{1}{2}}$ and $a_c = r_{\frac{1}{2}}$, $\varrho(x_1, h^3) = 2$ implies $\varrho(x_1, h^2) = 1$, i.e., one of the two established parties chooses *out* after the null history. Let this established party instead locate at x_1. After this history, the challenger's best response is $a_c = r_{\frac{1}{2}}$, so the established party's deviation is profitable. Suppose, instead, the equilibrium profile satisfies $\sum_{x \in X} \varrho(x) = 2$ and $X(h^2) = \{x_1, x_2\}$. Then $x_1 = 2r_{\frac{1}{2}} - x_2$, otherwise one of the parties wins with probability one. If $\sum_{x \in X(h^2)} \varrho(x, h^2) < 2$, a deviation to $\hat{a}_c = r_{\frac{1}{2}}$ is profitable. If $\sum_{x \in X(h^2)} \varrho(x, h^2) = 2$, the challenger strictly prefers an action contained in $\{x_1, x_2\}$ to the action *out*, since the challenger wins with probability .25 by locating at either platform. $\square$ *Claim 3.* There is no equilibrium profile satisfying $\sum_{x \in X} \varrho(x) = 0$. *Proof.* Trivial. $\square$ We conclude that if $p = .5$, an equilibrium profile satisfies $\sum_{x \in X} \varrho(x) = 3$. I next specify the strategy profile for the first class of equilibria. (i) After the null history, both established parties locate at $x_1 = r_{\frac{1}{2}}$. After a history h^2: (ii) if $X(h^2) = \{r_{\frac{1}{2}}\}$, $\varrho(r_{\frac{1}{2}}, h^2) \in \{1, 2\}$, $a_c = x_1$; (iii) if $X(h^2) = \{x_1, r_{\frac{1}{2}}\}$, $x_1 < r_{\frac{1}{2}}$, $\varrho(x_1, h^2) = \varrho(r_{\frac{1}{2}}, h^2) = 1$, $a_c = r_{\frac{1}{2}}$ if $x_1 \leq 2r_{\frac{1}{4}} - r_{\frac{1}{2}}$; otherwise $a_c = r_{\frac{1}{2}} + \epsilon$ satisfying $1 - F(r_{\frac{1}{2}} + \frac{1}{2}\epsilon) > F(.5(x_1 + r_{\frac{1}{2}})) > F(r_{\frac{1}{2}} + \frac{1}{2}\epsilon) - F(.5(x_1 + r_{\frac{1}{2}}))$. (iv) If $X(h^2) = \{r_{\frac{1}{2}}, x_2\}$, $x_2 > r_{\frac{1}{2}}$, $\varrho(r_{\frac{1}{2}}, h^2) = \varrho(x_2, h^2) = 1$, $a_c = r_{\frac{1}{2}}$ if $x_2 \geq 2r_{\frac{3}{4}} - r_{\frac{1}{2}}$; otherwise $a_c = r_{\frac{1}{2}} - \epsilon$ satisfying $F(r_{\frac{1}{2}} - \frac{1}{2}\epsilon) > 1 - F(.5(r_{\frac{1}{2}} + x_2)) > F(\frac{1}{2}(r_{\frac{1}{2}} + x_2)) - F(r_{\frac{1}{2}} - \frac{1}{2}\epsilon)$; (v) after any other history h^2, the challenger plays a best response. It is easily verified that this constitutes a subgame perfect Nash equilibrium. $\square$ Finally, I construct the second class of equilibria. (i) After the null history, one established party locates at $x_1 \in (\bar{x}, r_{\frac{1}{2}})$, and one established party locates at $x_2 = 2r_{\frac{1}{2}} - x_1$, where $\bar{x}$ is the infimum of the (non-empty) set of platforms $x < r_{\frac{1}{2}}$ that satisfies: for any $t \in (0, r_{\frac{1}{2}} - x)$, $F(r_{\frac{1}{2}} + t) - F(x + t) < \min\{F(x + t), 1 - F(r_{\frac{1}{2}} + t)\}$. (ii) If $X(h^2) = \{x_1, x_2\}$, $x_1 \in (\bar{x}, r_{\frac{1}{2}})$, $x_2 = 2r_{\frac{1}{2}} - x_1$, $\varrho(x_1, h^2) = \varrho(x_2, h^2) = 1$, then $a_c \in \{x_1, x_2\}$. (iii) After any other history, h^2, the challenger plays a best response. The challenger's strategy is trivially optimal. Optimality of the established parties' strategies follows from the following two claims. *Claim 4.* If $X(h^2) = \{x_1, x_2\}$, $x_1 \in (\bar{x}, r_{\frac{1}{2}})$, $x_2 < 2r_{\frac{1}{2}} - x_1$, $\varrho(x_1, h^2) = \varrho(x_2, h^2) = 1$, the challenger wins with probability 1 when it plays a best response. If $X(h^2) = \{x_1, x_2\}$, $x_2 \in (r_{\frac{1}{2}}, 2r_{\frac{1}{2}} - \bar{x})$, and $x_1 > 2r_{\frac{1}{2}} - x_2$, $\varrho(x_1, h^2) = \varrho(x_2, h^2) = 1$, the challenger wins with probability 1 when it plays a best response. *Proof.* I prove the first part. If the first condition holds, then $a_c \in (\max\{r_{\frac{1}{2}}, x_2\}, 2r_{\frac{1}{2}} - x_1)$ yields victory for the challenger with probability 1, and thus so does any best response after this history; the second case is similar. $\square$ *Claim 5.* If $X(h^2) = \{x_1, x_2\}$, $x_1 \in (\bar{x}, r_{\frac{1}{2}})$, and $x_2 > 2r_{\frac{1}{2}} - x_1$, $\varrho(x_1, h^2) = \varrho(x_2, h^2) = 1$, for any best response a_c the established party located at x_2 loses with probability 1. If $X(h^2) = \{x_1, x_2\}$, $x_2 \in (r_{\frac{1}{2}}, 2r_{\frac{1}{2}} - \bar{x})$,

$x_1 < 2r_{\frac{1}{2}} - x_2$ and $\varrho(x_1, h^2) = \varrho(x_2, h^2) = 1$, for any best response a_c the established party located at x_1 loses with probability 1. *Proof.* I prove the first part. If $a_c = out$, the claim is immediate. Suppose, instead, $a_c \in \mathbb{R}$ is a best response. If $a_c < 2r_{\frac{1}{2}} - x_2$ or $a_c > x_2$, the challenger loses in a second round contest with probability one. If $a_c = x_2$, the established party located at x_1 wins with probability one. Finally, if the challenger wins with positive probability by locating at $a_c = 2r_{\frac{1}{2}} - x_2$, then this location is strictly worse for the challenger than any location $a_c \in (2r_{\frac{1}{2}} - x_2, x_1)$. We conclude that $a_c \in \mathbb{R}$ is a best response only if $a_c \in (2r_{\frac{1}{2}} - x_2, x_2)$, in which case the established party located at x_2 does not win a majority in the first round, and loses with probability one in any second round contest. □

Proof of Proposition 1. I first show that if $p < .5$ and $q \leq \frac{1}{6}$, an equilibrium profile $\mathcal{X} = (X, \varrho, a_c)$ satisfies $\sum_{x \in X} \varrho(x) = 2$. It is trivially verified that there is no equilibrium profile satisfying $\sum_{x \in X} \varrho(x) = 0$. *Claim 1.* There is no equilibrium profile satisfying $\sum_{x \in X} \varrho(x) = 1$. *Proof.* If $a_c = out$, on the equilibrium path, a deviation to $a_c = r_{\frac{1}{2}}$ is profitable. Suppose, instead, $a_c \in \mathbb{R}$ on the equilibrium path, i.e., $\sum_{x \in X(h^2)} \varrho(x, h^2) = 0$. Let one established party deviate from *out* to $r_{\frac{1}{2}}$: the challenger's unique best response after this history is $a_c = r_{\frac{1}{2}}$, so the deviation by the established party is profitable. □ *Claim 2.* There is no equilibrium profile satisfying $\sum_{x \in X} \varrho(x) = 3$. *Proof.* Notice, first, that $\sum_{x \in X(h^3)} \varrho(x, h^3) = 3$ implies $\sum_{x \in X(h^2)} \varrho(x, h^2) = 2$, on the equilibrium path. (a) Suppose $X(h^2) = \{x_1\}$, on the equilibrium path. If $x_1 \neq r_{\frac{1}{2}}$, a deviation to $a_c = r_{\frac{1}{2}}$ is profitable; if $x_1 = r_{\frac{1}{2}}$, a deviation to $a_c = r_{\frac{1}{2}} + \epsilon$ for $\epsilon > 0$ satisfying $1 - F(r_{\frac{1}{2}} + \frac{1}{2}\epsilon) > \max\{p, \frac{1}{2}F(r_{\frac{1}{2}} + \frac{1}{2}\epsilon) + q\}$ is profitable. We conclude that $X(h^2) = \{x_1, x_2\}$, on the equilibrium path. (b) Suppose, further, $X(h^3) = \{x_1, x_2\}$, on the equilibrium path. It is straightforward to show that the challenger has no profitable deviation only if $x_1 \leq r_{\frac{1}{2}} \leq x_2$. Suppose $x_2 - r_{\frac{1}{2}} = r_{\frac{1}{2}} - x_1$; $X(h^3) = \{x_1, x_2\}$ implies $x_1 < r_{\frac{1}{2}} < x_2$, while $\sum_{x \in X(h^3)} \varrho(x, h^3) = 3$ implies $\min\{\varrho(x_1), \varrho(x_2)\} = 1$ and $\max\{\varrho(x_1), \varrho(x_2)\} = 2$; $p < .5$ and $q < .25$ implies a single party wins with probability one in the first round, a contradiction. Finally, a straightforward extension of the proof of Proposition A2 in Supplemental Appendix B can be used to rule out an equilibrium profile in which $\sum_{x \in X(h^3)} \varrho(x, h^3) = 3$, $X(h^3) = \{x_1, x_2\}$ on the equilibrium path, and in which $x_2 - r_{\frac{1}{2}} \neq r_{\frac{1}{2}} - x_1$. (c) Suppose, finally, $X(h^3) = \{x_1, x_2, x_3\}$, on the equilibrium path. It is straightforward to verify that the profile must satisfy (1) $x_1 < r_{\frac{1}{2}} < x_3$, (2) $a_c = x_2$ (otherwise the challenger could deviate slightly towards $r_{\frac{1}{2}}$), (3) either $x_1 < r_{\frac{1}{3}}$ or $x_3 > r_{\frac{2}{3}}$ (otherwise $a_2 = out$ is strictly preferred to $a_c = x_2$ for $x_2 \in (x_1, x_3)$) and (4) for any $z \in (x_1, x_3)$ either (a) $\min\{1 - F(.5(z + x_3)), F(.5(x_1 + z))\} \geq F(.5(z + x_3)) - F(.5(x_1 + z))$ or (b) $\min\{1 - F(.5(z + x_3)), F(.5(x_1 + z))\} < F(.5(z + x_3)) - F(.5(x_1 + z))$ and $\max\{1 - F(.5(z + x_3)), F(.5(x_1 + z))\} \geq \max\{p, F(.5(z + x_3)) - F(.5(x_1 + z)) + q\}$, otherwise the challenger could locate in (x_1, x_3) so as to win in the first or second round with probability one. Finally, I argue: (5) $r_{\frac{1}{2}} - x_1 = x_3 - r_{\frac{1}{2}}$. Suppose, to the contrary, that this property does not hold. I consider $r_{\frac{1}{2}} - x_1 < x_3 - r_{\frac{1}{2}}$, since the remaining case is similar. Let the established party that locates at $x_3 > 2r_{\frac{1}{2}} - x_1$ after the null history instead deviate to the location $2r_{\frac{1}{2}} - x_1$. By property (4) and $q < .25$, the challenger's strict best response after this deviation is *out*, and each established party wins with probability one half. We conclude $r_{\frac{1}{2}} - x_1 = x_3 - r_{\frac{1}{2}}$. To complete the argument, I rule out equilibrium profiles satisfying $X(h^3) = \{x_1, x_2, x_3\}$, on the equilibrium path, and in which either $x_1 < r_{\frac{1}{3}}$ or $x_2 > r_{\frac{2}{3}}$, i.e., I show that if property (3) holds, there is a profitable deviation for at least one agent. Suppose $x_1 < r_{\frac{1}{3}}$. Then, let the established party that is supposed to locate at x_1 deviate to $x_1 + \epsilon$, where $\epsilon > 0$ satisfies $1 - F(.5(x_1 + \epsilon + r_{\frac{1}{2}})) > \max\{p, \frac{1}{2}F(.5(x_1 + \epsilon + r_{\frac{1}{2}})) + q, F(x_1 + \epsilon) + q\}$. Such an $\epsilon > 0$ exists by continuity of F, the restriction $q \leq \frac{1}{6}$, and the supposition $x_1 < r_{\frac{1}{3}}$. After this deviation, $a_c = out$ is a strict best response, and thus we cannot have an equilibrium. The case $x_2 > r_{\frac{2}{3}}$ is similar. □

Next, I construct the strategy profile for the first class of equilibria, under the supposition $q \in [0, .25)$. (i) After the null history, one established party locates at $r_{\frac{1}{2}}$, and the other chooses *out*. The challenger's strategy is as follows. (ii) If $X(h^2) = \{r_{\frac{1}{2}}\}$, $\varrho(r_{\frac{1}{2}}, h^2) = 1$, $a_c = r_{\frac{1}{2}}$. (iii) If $X(h^2) = \{x_1, r_{\frac{1}{2}}\}$, $x_1 < r_{\frac{1}{2}}$, $\varrho(x_1, h^2) = \varrho(r_{\frac{1}{2}}, h^2) = 1$: if $\frac{1}{2} > \max\{\frac{1}{2} - F(\frac{1}{2}(x_1 + r_{\frac{1}{2}})), F(\frac{1}{2}(x_1 + r_{\frac{1}{2}}))\} + q$, $a_c = r_{\frac{1}{2}} + \epsilon$ satisfying $1 - F(r_{\frac{1}{2}} + \frac{1}{2}\epsilon) > \max\{p, F(r_{\frac{1}{2}} + \frac{1}{2}\epsilon) - F(\frac{1}{2}(x_1 + r_{\frac{1}{2}})) + q, F(\frac{1}{2}(x_1 + r_{\frac{1}{2}})) + q\}$; if this condition fails but $.25 < F(\frac{1}{2}(x_1 + r_{\frac{1}{2}}))$, $a_c = r_{\frac{1}{2}} + \epsilon$ satisfying $\epsilon < r_{\frac{1}{2}} - x_1$, $1 - F(r_{\frac{1}{2}} + \frac{1}{2}\epsilon) > F(\frac{1}{2}(x_1 + r_{\frac{1}{2}})) > F(r_{\frac{1}{2}} + \frac{1}{2}\epsilon) - F(\frac{1}{2}(x_1 + r_{\frac{1}{2}}))$; if both conditions fail but $F(\frac{1}{2}(x_1 + r_{\frac{1}{2}})) < \max\{p, \frac{1}{2}(1 - F(\frac{1}{2}(x_1 + r_{\frac{1}{2}})) + q\}$, $a_c = r_{\frac{1}{2}}$. Otherwise, $a_c = out$. (iv) If $X(h^2) = \{r_{\frac{1}{2}}, x_2\}$, $\varrho(r_{\frac{1}{2}}, h^2) = \varrho(x_2, h^2) = 1$, $x_2 > r_{\frac{1}{2}}$: if $\frac{1}{2} > \max\{F(\frac{1}{2}(r_{\frac{1}{2}} + x_2)) - \frac{1}{2}, 1 - F(\frac{1}{2}(r_{\frac{1}{2}} + x_2)\} + q$, $a_c = r_{\frac{1}{2}} - \epsilon$ satisfying $F(r_{\frac{1}{2}} - \frac{1}{2}\epsilon) > \max\{p, F(\frac{1}{2}(r_{\frac{1}{2}} + x_2)) - F(r_{\frac{1}{2}} - \frac{1}{2}\epsilon) + q, 1 - F(\frac{1}{2}(x_2 + r_{\frac{1}{2}})) + q\}$; if this condition fails but $F(\frac{1}{2}(r_{\frac{1}{2}} + x_2)) < .75$, $a_c = r_{\frac{1}{2}} - \epsilon$ satisfying $\epsilon < x_2 - r_{\frac{1}{2}}$, $F(r_{\frac{1}{2}} - \frac{1}{2}\epsilon) > 1 - F(\frac{1}{2}(r_{\frac{1}{2}} + x_2)) > F(\frac{1}{2}(r_{\frac{1}{2}} + x_2)) - F(r_{\frac{1}{2}} - \frac{1}{2}\epsilon)$; if both conditions fail but $1 - F(\frac{1}{2}(r_{\frac{1}{2}} + x_2)) < \max\{p, \frac{1}{2}(F(\frac{1}{2}(r_{\frac{1}{2}} + x_2)) + q\}$, $a_c = r_{\frac{1}{2}}$. Otherwise, $a_c = out$. (v) If $X(h^2) = \{r_{\frac{1}{2}}\}$, $\varrho(r_{\frac{1}{2}}, h^2) = 2$, $a_c = r_{\frac{1}{2}} + \epsilon$ satisfying $1 - F(r_{\frac{1}{2}} + \frac{1}{2}\epsilon) > \max\{p, \frac{1}{2}F(r_{\frac{1}{2}} + \frac{1}{2}\epsilon) + q\}$. (vi) If $X(h^2) = \{x_1\}$, $x_1 \neq r_{\frac{1}{2}}$, $a_c = r_{\frac{1}{2}}$. (vii) After any other history h^2, the challenger plays a best response. It is easy to verify that this is an equilibrium. $\square$

Finally, I construct the strategy profile for the second class of equilibria, under the supposition $q \in (0, .25)$. Let x^* denote the infimum of the (non-empty) set of platforms $x < r_{\frac{1}{2}}$ satisfying (1) $x \geq r_{\frac{1}{2}-q}$, $2r_{\frac{1}{2}} - x \leq r_{\frac{1}{2}+q}$, and (2) for any $t \in (0, r_{\frac{1}{2}} - x)$, $\min\{F(x_1 + t), 1 - F(r_{\frac{1}{2}} + t)\} > F(r_{\frac{1}{2}} + t) - F(x_1 + t)$. (i) After the null history, one established party locates at $x_1 \in (x^*, r_{\frac{1}{2}})$ and the other locates at $x_2 = 2r_{\frac{1}{2}} - x_1$. The challenger's strategy is as follows. (ii) If $X(h^2) = \{x_1\}$, $\varrho(x_1, h^2) = 1$, $x_1 \in (x^*, 2r_{\frac{1}{2}} - x^*)$, $a_c = r_{\frac{1}{2}}$. (iii) If $X(h^2) = \{x, y\}$ where $\varrho(x, h^2) = \varrho(y, h^2) = 1$: if $x \in (x^*, r_{\frac{1}{2}})$ and $y < 2r_{\frac{1}{2}} - x$, $a_c \in (\max\{r_{\frac{1}{2}}, y\}, 2r_{\frac{1}{2}} - x)$; if $x > 2r_{\frac{1}{2}} - y$ and $y \in (r_{\frac{1}{2}}, 2r_{\frac{1}{2}} - x^*)$, $a_c \in (2r_{\frac{1}{2}} - y, \min\{r_{\frac{1}{2}}, x\})$. (iv) If $X(h^2) = \{x_1, x_2\}$, $x_1 \in (x^*, r_{\frac{1}{2}})$, $x_2 = 2r_{\frac{1}{2}} - x_1$, $\varrho(x_1, h^2) = \varrho(x_2, h^2) = 1$, $a_c = out$. (v) After any other history, the challenger plays a best response. I show that this is an equilibrium. Points (ii) - (v) are straightforward. Point (i) follows from (ii)-(iv) and the next two claims. *Claim 3.* If $q > 0$: after any history $h^3 \in H^3$, no second round contest takes place if and only if a single party wins the election with probability one. *Proof.* No second round takes place if and only if there is a party i such that $v_i(\mathcal{X}(h^3)) \geq \max\{p, \max_{j \neq i} v_j(\mathcal{X}(h^3)) + q\}$; if $q > 0$ there is at most one party that satisfies this requirement. $\square$ *Claim 4.* If $X(h^2) = \{x_1, x_2\}$, $x_1 \in (x^*, r_{\frac{1}{2}})$, $x_2 > 2r_{\frac{1}{2}} - x_1$, $\varrho(x_1, h^2) = \varrho(x_2, h^2) = 1$, when the challenger plays a best response, the established party located at x_2 loses with probability one. If $X(h^2) = \{x_1, x_2\}$, $x_1 < 2r_{\frac{1}{2}} - x_2$, $x_2 \in (r_{\frac{1}{2}}, 2r_{\frac{1}{2}} - x^*)$, $\varrho(x_1, h^2) = \varrho(x_2, h^2) = 1$, when the challenger plays a best response, the established party located at x_1 loses with probability one. *Proof.* I prove the first claim. If $a_c = out$, the claim follows. If $a_c \in \mathbb{R}$ is a best response and the party located at x_2 wins with positive probability, *Claim 3* and $q > 0$ implies no party surpasses the threshold of exclusion in the first round. If the established party located at x_2 proceeds to a second round with positive probability, $a_c \in \mathbb{R}$ is a best response only if $a_c \in (2r_{\frac{1}{2}} - x_2, x_2)$. But this implies that the party located at x_2 loses in a second round with probability one. $\square$

Proof of Corollary 1. Existence of a best response for the challenger after any history $h^2 \in H^2$ in the setting with $n \geq 3$ established parties is a straightforward extension of Lemma 2. The extension of point 1 of Proposition 1 is trivial, so I focus on point 2. I consider $q \in (0, .25)$. Let x^* denote the infimum of the (non-empty) set of platforms $x < r_{\frac{1}{2}}$ satisfying $x \geq r_{\frac{1}{2}-q}$, $2r_{\frac{1}{2}} - x \leq r_{\frac{1}{2}+q}$, for any $t \in (0, r_{\frac{1}{2}} - x)$, $\min\{F(x), 1 - F(2r_{\frac{1}{2}} + x)\} > \{.25 + q, F(r_{\frac{1}{2}} + t) - F(x_1 + t) + q\}$. I specify the following strategy profile. (i) After the null history, one established party locates at $x_1 \in (x^*, r_{\frac{1}{2}})$, the other locates at $x_2 = 2r_{\frac{1}{2}} - x_1$, and the remaining established parties choose *out*. The challenger's strategy is as follows. (ii) If $X(h^2) = \{x_1\}$, $\varrho(x_1, h^2) = 1$, $x_1 \in (x^*, 2r_{\frac{1}{2}} - x^*)$, $a_c = r_{\frac{1}{2}}$. (iii) If $X(h^2) = \{x, y\}$ where $x \in (x^*, r_{\frac{1}{2}})$, $y < 2r_{\frac{1}{2}} - x$, $\varrho(x; h^2) = \varrho(y, h^2) = 1$, $a_c \in (\max\{r_{\frac{1}{2}}, y\}, 2r_{\frac{1}{2}} - x)$; if $y \in (r_{\frac{1}{2}}, 2r_{\frac{1}{2}} - x^*)$, $x > 2r_{\frac{1}{2}} - y$, $\varrho(x, h^2) = \varrho(y, h^2) = 1$, $a_c \in (2r_{\frac{1}{2}} - y, \min\{r_{\frac{1}{2}}, x\})$. (iv) If $X(h^2) = \{x_1, x_2\}$, $x_1 \in (x^*, r_{\frac{1}{2}})$, $x_2 = 2r_{\frac{1}{2}} - x_1$, $\varrho(x_1, h^2) = \varrho(x_2, h^2) = 1$,

$a_c = out$. (v) After any other history, $h^2 \in H^2$, the challenger plays a best response. I show that this is an equilibrium. Points (ii) - (v) and optimality of the platforms x_1 and x_2—conditional on the remaining $n-2$ established parties choosing *out*—follow from the proof of Proposition 1, so I focus on the strategy of the remaining $n-2$ established parties to choose *out*. The following claims establish that if one of these established parties instead locates at a platform in $\mathcal{R}$ after the null history, it wins the election with probability zero. *Claim 1.* If $X(h^2) = \{x_1, x_2, x_3\}$, $x_1 < x_2 \in (x^*, r_{\frac{1}{2}})$, $x_3 = 2r_{\frac{1}{2}} - x_2$, $\varrho(x_i, h^2) = 1$ for $i \in \{1,2,3\}$, whenever the challenger chooses a best response, the established party located at x_1 loses with probability one. If $X(h^2) = \{x_1, x_2, x_3\}$, $x_1 \in (x^*, r_{\frac{1}{2}})$, $x_2 = 2r_{\frac{1}{2}} - x_1 < x_3$, $\varrho(x_i, h^2) = 1$ for $i \in \{1,2,3\}$, whenever the challenger chooses a best response, the established party located at x_3 loses with probability one. *Proof.* Similar to Claim 4 in the proof of Proposition 1. □ *Claim 2.* If $X(h^2) = \{x_1, x_2, x_3\}$, $x_1 \in (x^*, r_{\frac{1}{2}})$, $x_1 < x_2 < x_3 = 2r_{\frac{1}{2}} - x_1$, $\varrho(x_i, h^2) = 1$ for $i \in \{1,2,3\}$, whenever the challenger chooses a best response, the established party located at x_2 after the history h^2 loses with probability one. *Proof.* If $a_c = out$, $x_1 > x^*$ implies that the party located at x_2 wins a strictly smaller vote share than each established party located at x_1 and x_3. If $a_c \in \mathbb{R}$, $a_c \in [x_1, x_3]$ is not a best response, since $x_1 > x^*$ implies that the challenger wins a vote share strictly less than each established party located at x_1 and x_3. By Claim 3 in the proof of Proposition 1, either $a_c < x_1$ or $a_c > x_3$ and the party located at x_2 winning with positive probability implies no party surpasses the threshold of exclusion in the first round, in which case the challenger loses in any second round contest, since it is located at the policy strictly furthest from $r_{\frac{1}{2}}$. Thus, $a_c \in \mathbb{R}$ is not a best response after this history. □ *Claim 3.* If $X(h^2) = \{x_1, x_2\}$, $x_1 \in (x^*, r_{\frac{1}{2}})$, $x_2 = 2r_{\frac{1}{2}} - x_1$, and $\min\{\varrho(x_1, h^2), \varrho(x_2, h^2)\} = 1$, $\max\{\varrho(x_1, h^2), \varrho(x_2, h^2)\} = 2$, whenever the challenger chooses a best response, it wins with probability one. *Proof.* I consider $X(h^2) = \{x_1, x_2\}$, $x_1 \in (x^*, r_{\frac{1}{2}})$, $x_2 = 2r_{\frac{1}{2}} - x_1$, and $\varrho(x_1, h^2) = 2$ and $\varrho(x_2, h^2) = 1$, since the argument for the case $\varrho(x_1, h^2) = 1$ and $\varrho(x_2, h^2) = 2$ is similar. Since $x_1 > x^*$, we have $1 - F(2r_{\frac{1}{2}} - x^*) > .25 + q$, so the challenger may locate at $a_c = x_2 + \epsilon$ for $\epsilon > 0$ satisfying $1 - F(x_2 + \frac{1}{2}\epsilon) > \max\left\{p, \frac{1}{4} + q, F(x_2 + \frac{1}{2}\epsilon) - \frac{1}{2} + q\right\}$, winning with probability one in the first round. Thus, if the challenger plays a best response, after this history it wins with probability one. □ Claims 1 through 3 imply that the strategy of the $n-2$ established parties that choose *out* is optimal. □

Proof of Lemma 1. *Claim 1.* In an equilibrium profile $\mathcal{X} = (X, \varrho, a_c)$, either (i) a single party wins a strict plurality and every other party ties for second place in the first round, or (ii) every party obtains the same share of the vote in the first round. *Proof.* The case $\sum_{x \in X} \varrho(x) \leq 2$ is trivial. If $\sum_{x \in X} \varrho(x) \geq 3$, the claim is true if there do not exist three parties i, j, k such that $v_i(\mathcal{X}) < \min\{v_j(\mathcal{X}), v_k(\mathcal{X})\}$. If such a triple exists, party i loses with probability 1. □ *Claim 2.* If a single party wins a strict plurality and all other parties that offer platforms tie for second place in the first round, then the single party which wins a strict plurality is located either at the platform furthest to the left (x_1) or the platform furthest to the right (x_k). *Proof.* Suppose the party that wins a strict plurality is located at platform $y \in \{x_2, ..., x_{k-1}\}$. If $y < r_{\frac{1}{2}}$, the party located at $x_1 < y$ loses with probability one. If $y \geq r_{\frac{1}{2}}$, the party located at x_k loses with probability one. □ *Claim 3.* If a single party wins a strict plurality and all other parties that offer platforms tie for second place in the first round, $x_1 < r_{\frac{1}{2}} < x_k$. *Proof.* Suppose $x_1 \geq r_{\frac{1}{2}}$. If a single party located at x_1 wins a plurality, it also wins a strict majority, and therefore wins with probability 1. If, instead, a single party located at x_k wins a strictly plurality, it either wins in the first round with probability one or with probability zero. In a second round, it loses with probability one against any other party: in all cases, at least one party loses the election with probability one; we conclude $x_1 < r_{\frac{1}{2}}$. The argument for $x_k > r_{\frac{1}{2}}$ is similar. □ *Claim 4.* If a single party wins a strict plurality in the first round, $r_{\frac{1}{2}} - x_1 = x_k - r_{\frac{1}{2}}$. *Proof.* Suppose $r_{\frac{1}{2}} - x_1 > x_k - r_{\frac{1}{2}}$.

If a single party located at x_1 wins a strictly plurality in the first round, it wins in the first round either with probability one, or with probability zero. If it wins with probability one, at least one other party loses with probability one. If it wins in the first round with probability zero, it loses in the second round to any other party, with probability one. The argument for the case $r_{\frac{1}{2}} - x_1 < x_k - r_{\frac{1}{2}}$ is similar. □ *Claim 5.* If a single party wins a strict plurality in the first round, it is an established party. *Proof.* Suppose the challenger wins a strict plurality in the first round. Then, by the previous steps, $a_c \in \{x_1, x_k\}$ and $\varrho(a_c) = 1$. Then, the challenger could deviate slightly towards $r_{\frac{1}{2}}$ and strictly increase its probability of winning by ensuring that it defeats any party located at $2r_{\frac{1}{2}} - a_c$ with probability one. □

SUPPLEMENTAL APPENDIX FOR ONLINE PUBLICATION.

Supplemental Appendix A: Properties of Equilibria with Three or More Platforms. I provide results on properties of equilibria with three or more platforms, for any number of platforms and any number of established parties, or challengers. Recall that $b_i(\mathcal{X}) = .5(x_i + x_{i+1})$ denotes the location of a voter that is indifferent between two adjacent platforms x_i and x_{i+1}, for $i \in \{1, ..., k-1\}$. To relate the following results to *Example 2* in the main text, expression (6) generalizes (3) for an arbitrary number of platforms and parties, whilst (7) gives corresponding formulae in the event that the plurality-winning established party locates at x_k, the largest platform, rather than x_1. The platform formulae (8) similarly generalize the platform formulae for *Example 2*, given in expression (4).

Proposition A1. Let $\mathcal{X} = (X, \varrho, a_c)$ denote an equilibrium profile with $X = \{x_1, ..., x_k\}$, $k \geq 3$.

1. If a single party is located at x_1 and wins a vote share $\alpha > \frac{1}{\sum_{x \in X} \varrho(x)}$, then:

$$b_i(\mathcal{X}) = F^{-1}\left(1 - \frac{\sum_{j=i+1}^{k} \varrho(x_j)}{\sum_{j=2}^{k} \varrho(x_j)}(1-\alpha)\right) \tag{6}$$

for $i \in \{1, ..., k-1\}$. If, instead, a single party is located at x_k and wins a vote share $\alpha > \frac{1}{\sum_{x \in X} \varrho(x)}$, then

$$b_i(\mathcal{X}) = F^{-1}\left(\frac{\sum_{j=1}^{i} \varrho(x_j)}{\sum_{j=1}^{k-1} \varrho(x_j)}(1-\alpha)\right) \tag{7}$$

for $i \in \{1, ..., k-1\}$. In both cases, the extreme platforms are symmetric around the median voter's most preferred policy, i.e., $x_k - r_{\frac{1}{2}} = r_{\frac{1}{2}} - x_1$, and:

1.a. if $|X|$ is an odd number, the location of platform $i \in \{1, ..., k-1\}$ is:

$$x_i = (-1)^i \left(\sum_{j=i}^{k-1} (-1)^j b_j(\mathcal{X}) - \mathbf{1}_{[i \geq 2]} \sum_{j=1}^{i-1} (-1)^j b_j(\mathcal{X}) - r_{\frac{1}{2}} \right), \tag{8}$$

1.b. if $|X|$ is an even number, the profile (α, X, ϱ) satisfies $r_{\frac{1}{2}} = \sum_{j=1}^{k-1} (-1)^{j-1} b_j(\mathcal{X})$.

2. If all parties that offer platforms tie for first place in the first round, then

$$b_i(\mathcal{X}) = F^{-1}\left(\frac{\sum_{j=1}^{i} \varrho(x_j)}{\sum_{j=1}^{k} \varrho(x_j)}\right) \tag{9}$$

for all $i \in \{1, ..., k-1\}$. Moreover: (2.i.) for almost any distribution $|X| \geq 4$ and $|X|$ is an even number implies $x_k - r_{\frac{1}{2}} \neq r_{\frac{1}{2}} - x_1$, and (2.ii.) regardless of whether $|X|$ is odd or even, $x_k - r_{\frac{1}{2}} > r_{\frac{1}{2}} - x_1$ implies $\varrho(x_k) \geq 2$, and $x_k - r_{\frac{1}{2}} < r_{\frac{1}{2}} - x_1$ implies $\varrho(x_1) \geq 2$.

Proof. Let $\mathcal{X} = (X, \varrho, y)$ be an equilibrium profile, $X = \{x_1, ..., x_k\}$. The derivation of $b_i(\mathcal{X})$ for each of the cases is straightforward.

1. Suppose a single party located at $x \in \{x_1, x_k\}$ wins vote share $\alpha > \frac{1}{\sum_{x_i \in X} \varrho(x_i)}$. Lemma 1 implies $x_k = 2r_{\frac{1}{2}} - x_1$. Since, $b_i(\mathcal{X}) \equiv .5(x_i + x_{i+1})$ for $i \in \{1, ..., k-1\}$, $x_{k-1} = 2b_{k-1}(\mathcal{X}) - x_k = 2b_{k-1}(\mathcal{X}) - (2r_{\frac{1}{2}} - x_1)$.

1.a. *Suppose* $|X|$ *is an odd number.* Proceeding recursively yields $x_1 = r_{\frac{1}{2}} - \sum_{j=1}^{k-1}(-1)^j b_j(\mathcal{X})$. If $k \geq 5$, for a platform x_i with $i \in \{3, ..., k-2\}$, i odd, we likewise obtain $x_i = 2(b_i(\mathcal{X}) - b_{i+1}(\mathcal{X})... - b_{k-1}(\mathcal{X})) + 2r_{\frac{1}{2}} - (r_{\frac{1}{2}} + \sum_{j=1}^{k-1}(-1)^{j-1}b_j(\mathcal{X}))$, i.e., $x_i = r_{\frac{1}{2}} + \sum_{j=1}^{i-1}(-1)^j b_j(\mathcal{X}) - \sum_{j=i}^{k-1}(-1)^j b_j(\mathcal{X})$. For any $k \geq 3$, the formulae for a platform x_i with $i \in \{2, ..., k-1\}$, i even, we similarly obtain $x_i = -r_{\frac{1}{2}} - \sum_{j=1}^{i-1}(-1)^j b_j(\mathcal{X}) + \sum_{j=i}^{k-1}(-1)^j b_j(\mathcal{X})$. Combining the cases yields expression (8).

1.b. *Suppose* $|X|$ *is an even number.* Similar steps as for the case of $|X|$ odd yield $x_1 = 2b_1(\mathcal{X}) - 2(b_2(\mathcal{X}) - b_3(\mathcal{X}) + ... + b_{k-1}) - x_k$; combining this with the supposition $x_k = 2r_{\frac{1}{2}} - x_k$ implies $r_{\frac{1}{2}} = \sum_{j=1}^{k-1}(-1)^{j-1}b_j(\mathcal{X})$.

2. Suppose, instead, that all parties that offer platforms tie for first place in the first round.

2.i. If $|X| \geq 4$, $|X|$ is even, and $x_k = 2r_{\frac{1}{2}} - x_1$, then a similar argument to 1.b. yields the condition $r_{\frac{1}{2}} = \sum_{j=1}^{k-1}(-1)^{j-1}b_j(\mathcal{X})$. Substituting the formulae for $b_i(\mathcal{X})$ into this expression, we obtain $F^{-1}\left(\frac{1}{2}\right) = \sum_{j=1}^{k-1}(-1)^{j-1}F^{-1}\left(\frac{\sum_{l=1}^{j}\varrho(x_j)}{\sum_{l=1}^{k-1}\varrho(x_l)}\right)$, which holds for almost no distribution.

2.ii For any $|X| \geq 4$, if $x_k - r_{\frac{1}{2}} > r_{\frac{1}{2}} - x_1$, at least two parties must locate at x_k, otherwise the single party located at that platform loses with probability one; the argument for $x_k - r_{\frac{1}{2}} < r_{\frac{1}{2}} - x_1$ is similar. $\square$

Supplemental Appendix B: Properties of Equilibria with Two Platforms. *Example 1* in the main text considers three established parties facing a single challenger, under the assumption that voters' ideal policies are uniformly distributed on the unit interval. In that example, I construct an equilibrium in which two established parties locate at $x_1 < .5$, and a single established party locates at platform $1 - x_1$; the challenger then locates at $1 - x_1$, and all parties win with probability .25. I show that the run-off thresholds p and q generate the restriction:

$$\max\left\{\frac{1}{3}\left(p + \frac{1}{2}\right), \frac{1+q}{4}\right\} \leq x_1 \leq \max\left\{p, \frac{1}{4} + q\right\}. \tag{10}$$

In this Supplemental Appendix, I examine in further detail some properties of equilibria with two platforms in which at least three parties win with positive probability. In all equilibria of this form, every party that offers a platform wins the same vote share in the first round, but all these parties do not necessarily enjoy the same probability of winning. The two platforms must always be located on opposite sides of the median (i.e., $x_1 < r_{\frac{1}{2}} < x_2$): I say that the equilibrium profile is *symmetric* if the platforms x_1 and x_2 are equidistant from the median voter's ideal point (that is, $x_2 - r_{\frac{1}{2}} = r_{\frac{1}{2}} - x_1$). Otherwise, I say that the profile is *asymmetric*, (i.e., $x_2 - r_{\frac{1}{2}} \neq r_{\frac{1}{2}} - x_1$). I consider properties of each class, in turn, restricting attention to run-off rules with $p \geq .25$; this encompasses all real-world settings.

Symmetric Profiles. If $p < .5$ and $q < .25$, in a symmetric two platform equilibrium in which at least three parties offer platforms, at least two parties—including, possibly the challenger—are positioned at each of the platforms, and the number of parties located at each platform is the same, i.e., $\varrho(x_1) = \varrho(x_2) = \varrho$.

Lemma A1. If $p < .5$ and $q < .25$, then an equilibrium profile $\mathcal{X}$ with two distinct platforms symmetrically located around the median has the following property: ϱ parties are located at each platform, i.e., $\varrho(x_1) = \varrho(x_2) = \varrho$. If $\varrho \geq 3$:

1. $a_c = x_1$ only if $\max\{r_{\frac{\varrho-1}{2\varrho}}, 2r_{\frac{1}{2}} - r_{\frac{\varrho}{2(\varrho-1)}}\} \leq \max\{r_p, r_{\frac{1}{2\varrho}+q}\}$.

2. $a_c = x_2$ only if $\max\{r_{\frac{\varrho-2}{2\varrho}}, 2r_{\frac{1}{2}} - r_{\frac{\varrho+1}{2\varrho}}\} \leq \max\{2r_{\frac{1}{2}} - r_{1-p}, 2r_{\frac{1}{2}} - r_{1-\frac{1}{2\varrho}-q}\}$.

Proof. Claim 1. If $p < .5$ and $q < .25$ and $\mathcal{X}$ is an equilibrium profile in which $|X| = 2$, $\varrho(x_1) + \varrho(x_2) \geq 3$ and $x_2 = 2r_{\frac{1}{2}} - x_1$, then $\varrho(x_1) = \varrho(x_2) \geq 2$.

Proof. Let $|X| = 2$, $r_{\frac{1}{2}} - x_1 = x_2 - r_{\frac{1}{2}}$, $\varrho(x_1) + \varrho(x_2) \geq 3$, $p < .5$ and $q < .25$. Suppose $\min\{\varrho(x_1), \varrho(x_2)\} = 1$. Then, $\varrho(x_1) + \varrho(x_2) \geq 3$ implies $\max\{\varrho(x_1), \varrho(x_2)\} \geq 2$. This implies which ever party is solely located at a platform wins in the first round with probability one. We conclude $\varrho(x_1) + \varrho(x_2) \geq 3$ implies $\min\{\varrho(x_1), \varrho(x_2)\} \geq 2$ and therefore every party that offers a platform wins the same vote share in the first round. Finally: $F\left(\frac{1}{2}(x_1 + x_2)\right) = \frac{\varrho(x_1)}{\varrho(x_1)+\varrho(x_2)}$ and $x_1 = 2r_{\frac{1}{2}} - x_2$ implies $\frac{\varrho(x_1)}{\varrho(x_1)+\varrho(x_2)} = \frac{1}{2}$. Thus, $\varrho(x_1) = \varrho(x_2)$. □

Next, let $\underline{x}(l, m)$ denote the infimum of the set of locations $x < r_{\frac{1}{2}}$ such that for all $t \in (0, r_{\frac{1}{2}} - x)$:

$$F\left(r_{\frac{1}{2}} + t\right) - F(x + t) \leq \max\left\{\frac{1}{l}\left(F(x + t)\right), \frac{1}{m}\left(1 - F\left(r_{\frac{1}{2}} + t\right)\right)\right\}. \tag{11}$$

Claim 2. $\underline{x}(\varrho - 1, \varrho) \geq \max\left\{r_{\frac{\varrho-1}{2\varrho}}, 2r_{\frac{1}{2}} - r_{\frac{\varrho}{2(\varrho-1)}}\right\}$, $\underline{x}(\varrho, \varrho - 1) \geq \max\left\{r_{\frac{\varrho}{2(\varrho+1)}}, 2r_{\frac{1}{2}} - r_{\frac{\varrho+1}{2\varrho}}\right\}$.

Proof. The platform $\underline{x}(\varrho - 1, \varrho)$ must satisfy:

$$\frac{1}{2} - F(\underline{x}(\varrho - 1, \varrho)) \leq \max\left\{\frac{1}{2\varrho}, \frac{1}{\varrho - 1}F(\underline{x}(\varrho - 1, \varrho))\right\}, \tag{12}$$

which simplifies to $\underline{x}(\varrho - 1, \varrho) \geq r_{\frac{\varrho-1}{2\varrho}}$. The platform must also satisfy:

$$F\left(2r_{\frac{1}{2}} - \underline{x}(\varrho - 1, \varrho)\right) - \frac{1}{2} \leq \max\left\{\frac{1}{2(\varrho - 1)}, \frac{1}{\varrho}\left(1 - F\left(2r_{\frac{1}{2}} - \underline{x}(\varrho - 1, \varrho)\right)\right)\right\}. \tag{13}$$

This simplifies to $\underline{x}(\varrho - 1, \varrho) \geq 2r_{\frac{1}{2}} - r_{\frac{\varrho}{2(\varrho-1)}}$. Thus, $\underline{x}(\varrho - 1, \varrho)$ satisfies:

$$\underline{x}(\varrho - 1, \varrho) \geq \max\left\{r_{\frac{\varrho-1}{2\varrho}}, 2r_{\frac{1}{2}} - r_{\frac{\varrho}{2(\varrho-1)}}\right\}. \tag{14}$$

Similarly, the platform $\underline{x}(\varrho, \varrho - 1)$ must satisfy:

$$F(2r_{\frac{1}{2}} - \underline{x}(\varrho, \varrho - 1)) - \frac{1}{2} \leq \max\left\{\frac{1}{2\varrho}, \frac{1}{\varrho - 1}\left(1 - F(2r_{\frac{1}{2}} - \underline{x}(\varrho, \varrho - 1))\right)\right\}, \tag{15}$$

which simplifies to $\underline{x}(\varrho, \varrho - 1) \geq 2r_{\frac{1}{2}} - r_{\frac{\varrho+1}{2\varrho}}$. It must further satisfy:

$$\frac{1}{2} - F(\underline{x}(\varrho, \varrho - 1)) \leq \max\left\{\frac{1}{\varrho}F(\underline{x}(\varrho, \varrho - 1)), \frac{1}{2(\varrho - 1)}\right\}. \tag{16}$$

This simplifies to $\underline{x}(\varrho, \varrho - 1) \geq r_{\frac{\varrho-2}{2(\varrho-1)}}$. Thus, $\underline{x}(\varrho, \varrho - 1)$ satisfies:

$$\underline{x}(\varrho, \varrho - 1) \geq \max\left\{r_{\frac{\varrho-2}{2(\varrho-1)}}, 2r_{\frac{1}{2}} - r_{\frac{\varrho+1}{2\varrho}}\right\}. \tag{17}$$

□

Claim 3. If $\mathcal{X}$ is an equilibrium profile in which $|X| = 2$ and the platforms are x_1 and $x_2 = 2r_{\frac{1}{2}} - x_1$, $\varrho(x_1) = \varrho(x_2) = \varrho \geq 3$, $a_c = x_1$ implies $x_1 \geq \underline{x}(\varrho - 1, \varrho)$ and $a_c = x_2$ implies $x_2 \geq \underline{x}(\varrho, \varrho - 1)$.

Proof. Suppose $a_c = x_1$ and $x_1 < \underline{x}(\varrho - 1, \varrho)$. Then, the challenger may deviate to a platform $a_c \in (x_1, x_2)$ and win the strictly greatest share of the vote in the first round. Since $\varrho \geq 3$, either a second round takes place in which case the challenger wins, or the challenger wins outright, in the first round. The case $a_c = x_2$ is similar. □

Claim 4. If $\mathcal{X}$ is an equilibrium profile in which $|X| = 2$ and the platforms are x_1 and $x_2 = 2r_{\frac{1}{2}} - x_1$, $\varrho(x_1) = \varrho(x_2) = \varrho \geq 3$, $a_c = x_1$ implies $x_1 \leq \max\{r_p, r_{\frac{1}{2\varrho}+q}\}$, and $a_c = x_2$ implies $x_1 \leq \max\{2r_{\frac{1}{2}} - r_{1-p}, 2r_{\frac{1}{2}} - r_{1-\frac{1}{2\varrho}-q}\}$.

Proof. A profitable deviation exists for the challenger in the subgame beginning after a history h^2 satisfying $X(h^2) = \{x_1, 2r_{\frac{1}{2}} - x_1\}$, $\varrho(x_1) = \varrho - 1$, $\varrho(x_2) = \varrho \geq 3$, if there is a platform $y < x_1$ satisfying $F(.5(y + x_1)) > \max\{p, \frac{1}{\varrho-1}(.5 - F(.5(y + x_1))) + q, \frac{1}{2\varrho} + q\}$, since locating at this platform ensures that the challenger wins with probability one in the first round. Notice that $\frac{1}{\varrho-1}(.5 - F(.5(y + x_1))) < \frac{1}{2\varrho}$ if and only if $F(.5(y + x_1)) > \frac{1}{2\varrho}$. By Claim 3, $x_1 \geq \underline{x}(\varrho, \varrho - 1)$ which implies $x_1 \geq r_{\frac{\varrho-1}{2\varrho}}$ by Claim 4, which implies $x_1 > r_{\frac{1}{2\varrho}}$ by the supposition $\varrho \geq 3$. Thus, for y sufficiently close to x_1, the condition $F(.5(y + x_1)) > \max\{p, \frac{1}{\varrho-1}(.5 - F(.5(y + x_1))) + q, \frac{1}{2\varrho} + q\}$ is equivalent to the condition $F(.5(y + x_1)) > \max\{p, \frac{1}{2\varrho} + q\}$. If $x_1 > \max\{p, \frac{1}{2\varrho} + q\}$, such a platform $y < x_1$ exists, and thus it is not a best response for the challenger to locate at the platform x_1. The proof for the case $a_c = x_2$ is similar. □

The Lemma is direct from these intermediate results. □

For example, if voter preferences are uniformly distributed on $[0, 1]$, an equilibrium profile in which $X = \{x_1, x_2\}$, $x_1 < r_{\frac{1}{2}} < x_2 = 2r_{\frac{1}{2}} - x_1$, $\max\{\varrho(x_1), \varrho(x_2)\} = 4$ and $a_c = x_1$ exists only if the run-off rule satisfies $\max\{p, \frac{1}{8} + q\} \geq \frac{3}{8}$. Finally, I consider symmetric two-platform profiles in which there are two parties located at each platform, and in which the challenger enters the contest. The following conditions generalize *Example 1* given in the main presentation.

Lemma A2. If $p < .5$ and $q < .25$, if an equilibrium profile $\mathcal{X}$ satisfies $X = \{x_1, x_2\}$, $r_{\frac{1}{2}} - x_1 = x_2 - r_{\frac{1}{2}}$ and $\max\{\varrho(x_1), \varrho(x_2)\} = 2$, then it also satisfies $\varrho(x_1) = \varrho(x_2) = 2$, and:

1. if $a_c = x_1$, then $r_{\frac{1}{4}} \leq x_1 \leq \max\{r_p, r_{\frac{1}{4}} + q\}$, and for any $t \in (0, r_{\frac{1}{2}} - x_1)$: either $F(r_{\frac{1}{2}} + t) - F(x_1 + t) < \frac{1}{2}(1 - F(r_{\frac{1}{2}} + t))$ or $F(x_1 + t) \geq \max\{p, F(r_{\frac{1}{2}} + t) - F(x_1 + t) + q\}$;
2. if $a_c = x_2$, then $2r_{\frac{1}{2}} - r_{\frac{3}{4}} \leq x_1 \leq \max\{2r_{\frac{1}{2}} - r_{1-p}, 2r_{\frac{1}{2}} - r_{\frac{3}{4}-q}\}$, and for any $t \in (0, r_{\frac{1}{2}} - x_1)$: either $F(r_{\frac{1}{2}} + t) - F(x_1 + t) < \frac{1}{2}F(x_1 + t)$ or $1 - F(r_{\frac{1}{2}} + t) \geq \max\{p, F(r_{\frac{1}{2}} + t) - F(x_1 + t) + q\}$; and,
3. if $n \geq 4$, $a_c \in \{x_1, x_2\}$ further implies that there exists $t \in (0, r_{\frac{1}{2}} - x_1)$ such that $F(r_{\frac{1}{2}} + t) - F(x_1 + t) > \max\{\frac{1}{2}(1 - F(r_{\frac{1}{2}} + t)), \frac{1}{2}F(\frac{1}{2}(x_1 + t))\}$.

Proof. I first prove point 1. If $x_1 > \max\{r_p, r_{\frac{1}{4}} + q\}$, the challenger may locate slightly to the right of x_1, and win with probability 1. The remaining conditions rule out a profitable deviation by the challenger to a platform on the interior of (x_1, x_2). Point 2 is similar. Finally, I prove point 3. Suppose $n \geq 4$ and $a_c = x_1$. To deter one of the $n - 3$ established parties that chooses out from instead locating at x_1, it must be the case that after this deviation, the challenger may locate at a platform on the interior (x_1, x_2) and win the strictly highest share of the vote, so that it wins with probability one. □

If the distribution of voter preferences has a density that is symmetric around the median, then points 1. and 2. are equivalent: for the uniform case, they specialize to condition (10), i.e., $\max\{\frac{1}{3}(p+\frac{1}{2}), \frac{1+q}{4}\} \leq x_1 \leq \max\{p, \frac{1}{4}+q\}$. Under the same distributional assumptions, point 3. is equivalent to $x_1 < \frac{1}{3}$.

Asymmetric Equilibria with Two Platforms. Depending on the distribution of voter ideal policies, F, equilibria with two distinct platforms that are asymmetrically located around the median voter's most preferred policy may exist, i.e., equilibria satisfying $r_{\frac{1}{2}} - x_1 \neq x_2 - r_{\frac{1}{2}}$. When they do exist, some broad points about the performance of the challenger can be given.

Proposition A2. In every asymmetric equilibrium with two platforms in which the challenger wins the election with positive probability:

1. the challenger is (weakly) less likely to win than any established party which offers a platform; and,
2. at least three established parties are strictly more than three times as likely to win the election as the challenger.

Proof. I first claim that if $x_1 \neq 2r_{\frac{1}{2}} - x_2$, then $\varrho(x_1) \neq \varrho(x_2)$. Suppose, to the contrary, $\varrho(x_1) = \varrho(x_2)$. I consider $r_{\frac{1}{2}} - x_1 < x_2 - r_{\frac{1}{2}}$, since the reverse strict inequality follows a similar argument. If $\varrho(x_1) = \varrho(x_2) \geq 2$, each party located at x_2 loses in the first round with probability one, since the (at least) two parties located at x_1 win a strictly higher share of the vote in the first round than any of the parties located at x_2. If $\varrho(x_1) = \varrho(x_2) = 1$, the party located at x_1 wins a majority in the first round. We conclude: $\varrho(x_1) \neq \varrho(x_2)$.

Next, I claim that if $x_1 \neq 2r_{\frac{1}{2}} - x_2$ and $a_c \in \{x_1, x_2\}$, then $\min\{\varrho(x_1), \varrho(x_2)\} \geq 2$. Suppose, to the contrary, $\min\{\varrho(x_1), \varrho(x_2)\} = 1$. Suppose, first, $r_{\frac{1}{2}} - x_1 < x_2 - r_{\frac{1}{2}}$. If $\varrho(x_1) = 1$, the party located at x_1 wins a majority in the first round. Thus $\varrho(x_1) \geq 2$. Suppose $\varrho(x_2) = 1$. Then, the party located at x_2 must be an established party: if $a_c = x_2$ on the equilibrium path, the challenger could instead locate at $r_{\frac{1}{2}}$ and win with probability one in the first round. Suppose, first, $\varrho(x_1) \geq 3$. Then, either (1) $F(x_1) > \frac{1}{\varrho(x_1)+1} > \frac{1}{\varrho(x_1)-1}(\frac{\varrho(x_1)}{\varrho(x_1)+1} - F(x_1))$ or (2) $\frac{\varrho(x_1)}{\varrho(x_1)+1} - F(x_1) > \max\left\{\frac{1}{\varrho(x_1)-1}F(x), \frac{1}{\varrho(x_1)+1}\right\}$. This implies that the challenger may deviate to some platform slightly to the left or slightly to the right of x_1 and win with probability one. So, $\min\{\varrho(x_1), \varrho(x_2)\} = 1$, $a_c \in \{x_1, x_2\}$ and $r_{\frac{1}{2}} - x_1 < x_2 - r_{\frac{1}{2}}$ implies $\varrho(x_1) = 2$ and $\varrho(x_2) = 1$. Further, each of the three parties must win with probability $\frac{1}{3}$ in the first round, since the single established party located at x_2 loses with probability one in a second round contest. This implies $F(.5(x_1+x_2)) = \frac{2}{3}$. This implies $x_1 = r_{\frac{1}{3}}$, since otherwise the challenger could deviate to a platform $a_c < x_1$ or $a_c > x_1$ and win the election with probability one. $F(.5(x_1+x_2)) = \frac{2}{3}$ in turn yields $x_2 = 2r_{\frac{2}{3}} - r_{\frac{1}{3}}$, $p \leq \frac{1}{3}$ and $q = 0$. We conclude $a_c = x_1 = r_{\frac{1}{3}}$. The challenger does not strictly prefer some action $a_c \in (r_{\frac{1}{3}}, 2r_{\frac{2}{3}} - r_{\frac{1}{3}})$ only if for all $t \in (0, r_{\frac{2}{3}} - r_{\frac{1}{3}})$: $\max\{F(r_{\frac{1}{3}}+t), 1 - F(r_{\frac{2}{3}}+t)\} > F(r_{\frac{2}{3}}+t) - F(r_{\frac{1}{3}}+t)$. Let the established party which is supposed to locate at $x_2 = 2r_{\frac{2}{3}} - r_{\frac{1}{3}}$ deviate to $x_2' = 2r_{\frac{1}{2}} - r_{\frac{1}{3}}$. I claim after this history, $a_c = out$ is the challenger's unique best response. To see this note, that $a_c \leq x_1$ and $a_c \geq x_2$ is not optimal, since one of the established parties wins with probability one in the first round, and by construction, the challenger loses in the first round with probability one by any choice of location $a_c \in (x_1, x_2)$. Thus, the deviating established party raises its probability of victory to .5. We conclude that $r_{\frac{1}{2}} - x_1 < x_2 - r_{\frac{1}{2}}$ and $a_c \in \{x_1, x_2\}$ implies $\min\{\varrho(x_1), \varrho(x_2)\} \geq 2$. A similar argument deals with the case of $r_{\frac{1}{2}} - x_1 > x_2 - r_{\frac{1}{2}}$. By Lemma 1, this implies that all parties which offer a platform tie for first place in the first round of the contest.

Finally, I show that if $a_c \in \{x_1, x_2\}$, then $r_{\frac{1}{2}} - x_1 > x_2 - r_{\frac{1}{2}}$ implies $a_c = x_1$, and $r_{\frac{1}{2}} - x_1 < x_2 - r_{\frac{1}{2}}$ implies $a_c = x_2$. Suppose $r_{\frac{1}{2}} - x_1 > x_2 - r_{\frac{1}{2}}$. Since all players tie in the first round, we obtain $F(.5(x_1+x_2)) =$

$\frac{\varrho(x_1)}{\varrho(x_1)+\varrho(x_2)} < \frac{1}{2}$, i.e., $\varrho(x_2) > \varrho(x_1)$. I show $a_c \in \{x_1, x_2\}$ implies $a_c = x_1$. Suppose, to the contrary, $a_c = x_2$. If both:

$$1 - F(x_2) > \frac{1}{\varrho(x_2) + \varrho(x_1)} > \frac{1}{\varrho(x_2) - 1}\left(F(x_2) - \frac{\varrho(x_1)}{\varrho(x_1) + \varrho(x_2)}\right), \tag{18}$$

which is equivalent to $F(x_2) < \frac{\varrho(x_2)+\varrho(x_1)-1}{\varrho(x_2)+\varrho(x_1)}$, then there exists a policy slightly to the right of x_2 to which the challenger can deviate, such that it wins in the first round, or proceeds to the second round and defeats another party located at $x_1 < 2r_{\frac{1}{2}} - x_2$ with probability one. Alternatively, if:

$$F(x_2) - \frac{\varrho(x_1)}{\varrho(x_1) + \varrho(x_2)} > \max\left\{\frac{1}{\varrho(x_2) - 1}(1 - F(x_2)), \frac{1}{\varrho(x_2) + \varrho(x_1)}\right\}, \tag{19}$$

which is equivalent to $F(x_2) > \frac{\varrho(x_1)+1}{\varrho(x_1)+\varrho(x_2)}$, then a similar deviation by the challenger slightly to the left of x_2 can be made. If $\varrho(x_2) \geq 3$, we have $\frac{\varrho(x_1)+1}{\varrho(x_1)+\varrho(x_2)} < \frac{\varrho(x_1)+\varrho(x_2)-1}{\varrho(x_1)+\varrho(x_2)}$, and we are done. But we must have $\varrho(x_2) \geq 3$ since $\varrho(x_2) > \varrho(x_1)$ by supposition, and I have already shown $\min\{\varrho(x_1), \varrho(x_2)\} \geq 2$. Thus, $r_{\frac{1}{2}} - x_1 > x_2 - r_{\frac{1}{2}}$ and $a_c \in \{x_1, x_2\}$ implies $a_c = x_1$, as was to be shown. The argument is easily extended to show $r_{\frac{1}{2}} - x_1 < x_2 - r_{\frac{1}{2}}$ implies $a_c = x_2$.

We conclude that, if the challenger locates at a platform $a_c \in \{x_1, x_2\}$, the ratio of (1) the probability of winning of an established party located at $\{x_1, x_2\} \setminus \{a_c\}$ to (2) the challenger's probability of winning is:

$$\frac{\frac{1}{2}(\max\{\varrho(x_1), \varrho(x_2)\} - 1) + \min\{\varrho(x_1), \varrho(x_2)\}}{\frac{1}{2}(\min\{\varrho(x_1), \varrho(x_2)\} - 1)} \geq 3\frac{\min\{\varrho(x_1), \varrho(x_2)\}}{\min\{\varrho(x_1), \varrho(x_2)\} - 1} > 3 \tag{20}$$

And since $\max\{\varrho(x_1), \varrho(x_2)\} > \min\{\varrho(x_1), \varrho(x_2)\} \geq 2$, at least three established parties are strictly more than three times as likely to win the contest as the challenger. □

Supplemental Appendix C: Results for a Setting with Two Challengers. I first prove Proposition 2, then extend Example 3 to a more general family of distribution functions.

Proof of Proposition 2. First, I construct the strategy profile. Let x^* denote the infimum of the set of platforms $x < r_{\frac{1}{2}}$ satisfying (1) $x \geq r_{\frac{1}{2}-q}$, $2r_{\frac{1}{2}} - x \leq r_{\frac{1}{2}+q}$, and (2) for any $t \in [0, r_{\frac{1}{2}} - x]$, $\min\{F(x), 1 - F(2r_{\frac{1}{2}} + x)\} > \max\{.25 + q, F(r_{\frac{1}{2}} + t) - F(x_1 + t) + q\}$. The strategy profile is as follows. (i) After the null history, one established party locates at $x_1 \in (x^*, r_{\frac{1}{2}})$ and one established party locates at $x_2 = 2r_{\frac{1}{2}} - x_1$. (ii) After a history h^2 such that $X(h^2) = \{x_1, x_2\}$, $x_1 \in (x^*, r_{\frac{1}{2}})$, $x_2 = 2r_{\frac{1}{2}} - x_1$, $\varrho(x_1, h^2) = \varrho(x_2, h^2) = 1$, the first challenger chooses *out*. (iii) If $X(h^2) = \{x\}$, where $x \in (x^*, r_{\frac{1}{2}}) \cup (r_{\frac{1}{2}}, 2r_{\frac{1}{2}} - x^*)$, $\varrho(x, h^2) = 1$, the first challenger locates at $2r_{\frac{1}{2}} - x$. (iv) If $X(h^2) = \{x_1, x_2\}$, $x_2 \in (r_{\frac{1}{2}}, 2r_{\frac{1}{2}} - x^*)$, $x_1 \in (2r_{\frac{1}{2}} - x_2, x_2)$, $\varrho(x_1, h^2) = \varrho(x_2, h^2) = 1$, the first challenger locates at $2r_{\frac{1}{2}} - x_2$. If $X(h^2) = \{x_1, x_2\}$ where $x_1 \in (x^*, r_{\frac{1}{2}})$, $x_2 \in (x_1, 2r_{\frac{1}{2}} - x_1)$, $\varrho(x_1, h^2) = \varrho(x_2, h^2) = 1$, the first challenger locates at $2r_{\frac{1}{2}} - x_1$. (v) After any other history h^2, the first challenger plays a best response. (vi) After any history h^3, the second challenger plays a best response. I show that this strategy is an equilibrium. A straightforward extension of Lemma 2 can be used to show that after any history $h^2 \in H^2$, an equilibrium exists. I first show that the first challenger's action specified in (ii) is optimal.

Claim 1. Under the strategy profile specified above: in an equilibrium of the subgame beginning after history h^2, where $X(h^2) = \{x_1, x_2\}$, $x_1 \in (x^*, r_{\frac{1}{2}})$, $x_2 = 2r_{\frac{1}{2}} - x_1$, $\varrho(x_1, h^2) = \varrho(x_2, h^2) = 1$, it is a best response for the first challenger to choose *out*.

Proof. Let $a_1 \in \mathbb{R} \cup \{out\}$ denote the action of the first challenger, taken after a history h^2, and $a_2 \in \mathbb{R} \cup \{out\}$ denote the action of the second challenger, taken after a history h^3. (a) Suppose, after h^2, $a_1 < x_1$ or $a_1 > x_2$.

The first challenger loses with probability one unless, after this history, $a_2 \in \mathbb{R}$. By Claim 3 of the proof of Proposition 1, $a_1 \in \mathbb{R}$, $a_2 \in \mathbb{R}$ and $q > 0$ imply no party surpasses the threshold of exclusion in the first round. And $a_1 \in \mathbb{R}$ is a best response only if the first challenger proceeds to the second round with positive probability. Thus, if $a_1 < x_1$ is a best response, $x_2 = 2r_{\frac{1}{2}} - x_1$, and $\varrho(a_1, h^3) = \varrho(x_1, h^3) = \varrho(x_2, h^3) = 1$, the action $a_2 \in \mathbb{R}$ is also best response only if $a_2 \in (a_1, 2r_{\frac{1}{2}} - a_1)$. But this implies that the first challenger loses in any second round contest. A similar argument rules out $a_1 > x_2$. (b) Suppose, after h^2, $a_1 \in \{x_1, x_2\}$. If $a_1 = x_1$, the second challenger wins with probability one by locating at $a_2 = x_2 + \epsilon$, $\epsilon > 0$ satisfying $1 - F(x_2 + \frac{1}{2}\epsilon) > \max\left\{p, \frac{1}{4} + q, F(x_2 + \frac{1}{2}\epsilon) - \frac{1}{2} + q\right\}$, which exists by continuity of $F(\cdot)$ and construction of x^*. Thus, the second challenger wins with probability one whenever it plays a best response after this history. A similar argument rules out a best response $a_1 = x_2$. (c) Finally, suppose that after h^2, $a_1 \in (x_1, x_2)$. The first challenger loses with probability one if the second challenger chooses $a_2 = out$, by construction of x^*. Suppose, to the contrary, $a_2 \in \mathbb{R}$, after this history. (c.1) By construction of x^*, $a_2 = out$ is strictly preferred to $a_2 \leq x_1$ or $a_2 \geq x_2$. (c.2) If $a_2 \in (x_1, x_2)$, $x_1 > x^*$ implies the second challenger loses the election with probability one in the first round. Thus, after this history, $a_2 = out$ is a unique best response for the second challenger, so the first challenger located at $a_1 \in (x_1, x_2)$ loses in the first round with probability one. We conclude that $a_1 = out$ is the unique best response after a history h^2 in which $X(h^2) = \{x_1, x_2\}$, $x_1 \in (x^*, r_{\frac{1}{2}})$, $x_2 = 2r_{\frac{1}{2}} - x_1$, $\varrho(x_1) = \varrho(x_2) = 1$. □

This shows that the challenger's action specified in (ii) is optimal.

Claim 2. In the subgame after a history h^3 with $X(h^3) = \{x_1, x_2\}$, $x_1 \in (x^*, r_{\frac{1}{2}})$, $x_2 = 2r_{\frac{1}{2}} - x_1$, $\varrho(x_1, h^3) = \varrho(x_2, h^3) = 1$, $a_2 = out$ is a strict best response.

Proof. Immediate, by construction of x^*. □

Verifying (iii) and (iv) follow straightforwardly from the previous Claim, and (v) and (vi) are trivial. It remains to verify point (i). The previous two Claims imply that, in every equilibrium after the history $X(h^2) = \{x_1, x_2\}$, $x_1 = x^*$, $x_2 = 2r_{\frac{1}{2}} - x_1$, each established party wins the election with probability one half. Thus, it is sufficient to show:

Claim 3. Under the strategy profile specified above: in the equilibrium of the subgame after a history h^2 with two platforms x and y, $y \in (r_{\frac{1}{2}}, 2r_{\frac{1}{2}} - x^*)$, $x \in \mathbb{R} \setminus \{2r_{\frac{1}{2}} - y\}$, $\varrho(x, h^2) = \varrho(y, h^2) = 1$, the party located at x after history h^2 wins with probability weakly less than one half. Under the strategy profile specified above: in the equilibrium of the subgame after a history h^2 with two platforms x and y, $x \in (x^*, r_{\frac{1}{2}})$, $y \in \mathbb{R} \setminus \{2r_{\frac{1}{2}} - x\}$, $\varrho(x, h^2) = \varrho(y, h^2) = 1$, the party located at y after history h^2 wins with probability weakly less than one half.

Proof. I establish the first part of the claim, since the second part is similar. (a) If $x \in (2r_{\frac{1}{2}} - y, y)$, points (iv) and (vi) of the strategy profile imply that the party located at x loses with probability one. (b) Suppose $x < 2r_{\frac{1}{2}} - y$. Let $a_1 \in \mathbb{R} \cup \{out\}$ denote the action of the first challenger, and $a_2 \in \mathbb{R} \cup \{out\}$ denote the action of the second challenger. (b.1) If $a_1 = a_2 = out$, the claim is true. (b.2) Suppose one challenger $i \in \{1, 2\}$ chooses $a_i \in \mathbb{R}$ and the other chooses $a_j = out$. By Claim 3 of Proposition 1, $q > 0$, $a_i \in \mathbb{R}$ is a best response and the party at x wins with positive probability only if no party wins in the first round with positive probability. Since $x < 2r_{\frac{1}{2}} - y < r_{\frac{1}{2}} < y$, this implies $r_{\frac{1}{2}} - x \leq |a_i - r_{\frac{1}{2}}|$, and $|a_i - r_{\frac{1}{2}}| \leq r_{\frac{1}{2}} - x$; thus, $r_{\frac{1}{2}} - x = |a_i - r_{\frac{1}{2}}|$, so the probability that the party located at x wins in the second round—and thus in the election—can be at most one half. (b.3) Suppose, finally, $a_1 \in \mathbb{R}$ and $a_2 \in \mathbb{R}$, after the history h^2.

Claim 3 of Proposition 1 and $q > 0$ implies each challenger is choosing a best response only if no party wins in the first round, and each challenger competes in a second round contest with positive probability. (b.3.i) If $\max_{i\in\{1,2\}} |a_i - r_{\frac{1}{2}}| \leq r_{\frac{1}{2}} - x$, the claim is true. (b.3.ii) If $\max_{i\in\{1,2\}} |a_i - r_{\frac{1}{2}}| > r_{\frac{1}{2}} - x$, the necessity of a second round contest and $r_{\frac{1}{2}} - x > y - r_{\frac{1}{2}}$ implies $|a_1 - r_{\frac{1}{2}}| = |a_2 - r_{\frac{1}{2}}|$, and thus either $a_1 < x$ or $a_1 > 2r_{\frac{1}{2}} - x$. (b.3.ii.a) Suppose $a_1 \neq a_2$. If $a_1 < x$, then $|a_1 - r_{\frac{1}{2}}| = |a_2 - r_{\frac{1}{2}}|$ implies $a_2 = 2r_{\frac{1}{2}} - a_1 > 2r_{\frac{1}{2}} - x > y$. But if the challenger located at a_1 proceeds to a second round with positive probability, and the second challenger wins with positive probability by locating at $a_2 = 2r_{\frac{1}{2}} - a_1$, a location $\hat{a}_2 \in (y, 2r_{\frac{1}{2}} - x)$ is strictly preferred by the second challenger to the location $2r_{\frac{1}{2}} - a_1$. A similar argument rules out $a_1 \neq a_2$, $a_1 > 2r_{\frac{1}{2}} - x$. (b.3.ii.b) Suppose $a_1 = a_2 \equiv a$. We have shown either $a < x$ or $a > 2r_{\frac{1}{2}} - x$. Suppose $a < x$; then $.5F(.5(a + x)) < .25$ by $a_1 < x < r_{\frac{1}{2}}$; since there are only four parties locating at platforms in $\mathbb{R}$, at least one of the two parties located at either x or y—and thus strictly closer to $r_{\frac{1}{2}}$—wins a strictly higher first-round vote share. This implies that the challengers located at platform a lose in a second round contest with probability one. The case $a > 2r_{\frac{1}{2}} - x$ is similar. This completes the argument for the case (b), i.e., $x < 2r_{\frac{1}{2}} - x$. (c) Arguments similar to (b) establish that after a history h^2 satisfying $X(h^2) = \{y, x\}$, $x > y$, $y \in (r_{\frac{1}{2}}, 2r_{\frac{1}{2}} - x^*)$, $\varrho(x, h^2) = \varrho(y, h^2) = 1$, the established party located at y wins a vote share weakly less than one half. (d) Finally, if $x = y$, $\varrho(x_2) \geq 2$ implies each party located at y wins with probability weakly less than one half in *every* equilibrium of the subgame after this history h^2. □ This implies that there is no profitable deviation for either established party from the actions specified in (i). This concludes the proof of existence. □

I next prove the second part of the Proposition: that if $p < .5$ and $q \in (0, .25)$, there is no equilibrium profile $\mathcal{X}(h^4) = (X(h^4), \varrho(\mathcal{X}(h^4)), a_1, a_2)$ in which $|X(h^4)| = 1$. I establish two intermediate results. The first states that in every equilibrium of the subgame beginning after a history h^2 in which a single established party locates at $r_{\frac{1}{2}}$ and the other established party chooses *out*, the first challenger subsequently chooses *out* and the second challenger locates at $r_{\frac{1}{2}}$. The proof is a simple extension of the proof of Proposition 1, and omitted.

Claim 4. In a setting with two challengers, $p < .5$ and $q < .25$, in every subgame perfect equilibrium after a history $X(h^2) = \{r_{\frac{1}{2}}\}$, $\varrho(r_{\frac{1}{2}}) = 1$, the first challenger chooses *out* and the second challenger locates at $r_{\frac{1}{2}}$.

We conclude that in any equilibrium, at least one of the two established parties offers a policy in $\mathbb{R}$. The next Claim shows that in any equilibrium with a single platform, (1) this platform is the median voter's most preferred policy, (2) a single established party locates at this platform, (3) the second challenger also locates at this platform and (4) the first challenger and the remaining established party each choose *out*.

Claim 5. If $p < .5$ and $q \in (0, .25)$, in the setting with two challengers, an equilibrium profile $\mathcal{X} = (X, \varrho, a_1, a_2)$ satisfies $X = \{x_1\}$ only if $x_1 = r_{\frac{1}{2}}$, $\varrho(r_{\frac{1}{2}}; h^2) = 1$, $a_1 = out$ and $a_2 = r_{\frac{1}{2}}$.

Proof. By Claim 4, there is no equilibrium in which $X(h^2) = \emptyset$, since one established party could instead locate at $r_{\frac{1}{2}}$ and win with probability one half. Thus, we may restrict attention to equilibria in which, on the equilibrium path, $|X(h^2)| = 1$, $x_1 \in \mathbb{R}$, $\varrho(x_1) \geq 1$. I next consider properties of the history h^3—the history after which the second challenger makes its location-entry choice—that must be satisfied on the path of an equilibrium in which $|X(h^2)| = 1$ and $\varrho(x_1, h^2) \geq 1$. (1) Suppose $X(h^3) = \{x_1\}$, $x_1 \neq r_{\frac{1}{2}}$. Then the second challenger wins with probability one by locating at $r_{\frac{1}{2}}$, so that any party located at x_1 loses with probability one when the second challenger plays a best response, after this history h^3. (2) Suppose $X(h^3) = \{r_{\frac{1}{2}}\}$. The previous Claim implies $\varrho(r_{\frac{1}{2}}, h^2) \geq 1$. If, $\varrho(r_{\frac{1}{2}}, h^3) \geq 2$, the second challenger may locate at $r_{\frac{1}{2}} + \epsilon$ for

$\epsilon > 0$ such that $1 - F(r_{\frac{1}{2}} + \frac{1}{2}\epsilon) > \max\{.5F(r_{\frac{1}{2}} + \frac{1}{2}\epsilon) + q, p\}$ and win with probability one, since $q < .25$. We conclude $\varrho(r_{\frac{1}{2}}, h^3) = 1$, and therefore $\varrho(r_{\frac{1}{2}}, h^2) = 1$. This implies $a_1 = out$, and $a_2 = r_{\frac{1}{2}}$. □

The previous two claims imply that an equilibrium profile $\mathcal{X}(h^4) \equiv \mathcal{X}$ satisfies $|X| = 1$ only if $X = \{r_{\frac{1}{2}}\}$, $\varrho(r_{\frac{1}{2}}, h^2) = 1$, $a_1 = out$, and $a_2 = r_{\frac{1}{2}}$. A strategy profile that is an equilibrium supporting this profile must specify that, after the null history, one of the established parties chooses *out*, and the other locates at $r_{\frac{1}{2}}$. Let the established party that chooses *out* under the strategy profile instead locate at x satisfying $r_{\frac{1}{2}} > x > \max\{r_p, r_{\frac{1}{2}-q}\}$, $2r_{\frac{1}{2}} - x < \min\{r_{1-p}, r_{\frac{1}{2}+q}\}$, and $F(2r_{\frac{1}{2}} - x) - F(x) + q < \min\{F(x), 1 - F(2r_{\frac{1}{2}} - x)\}$. I show that on the path of every equilibrium after the history h^2 such that $X(h^2) = \{x, r_{\frac{1}{2}}\}$, $\varrho(r_{\frac{1}{2}}, h^2) = \varrho(x, h^2) = 1$, and x satisfies these constraints, the established party located at x wins with probability .5, implying that the deviation is strictly profitable, and therefore this strategy profile cannot be an equilibrium.

Claim 6. In every equilibrium of the subgame after a history h^2 with $X(h^2) = \{x, r_{\frac{1}{2}}\}$ where x satisfies $r_{\frac{1}{2}} > x > \max\{r_p, r_{\frac{1}{2}-q}\}$, $2r_{\frac{1}{2}} - x < \min\{r_{1-p}, r_{\frac{1}{2}+q}\}$, and $F(2r_{\frac{1}{2}} - x) - F(x) + q < \min\{F(x), 1 - F(2r_{\frac{1}{2}} - x)\}$, $\varrho(r_{\frac{1}{2}}, h^2) = \varrho(x, h^2) = 1$: $a_1 = 2r_{\frac{1}{2}} - x$, and $a_2 = out$ on the equilibrium path.

Proof. (a) If $a_1 = 2r_{\frac{1}{2}} - x$, it is easy to verify that $a_2 = out$ is a strict best response. We therefore consider alternative locations for the first challenger, $\hat{a}_1 \in \mathbb{R} \setminus \{2r_{\frac{1}{2}} - x\}$. It is sufficient to show that in every equilibrium of the subgame after a history in which $\hat{a}_1$ is chosen, the party located at $\hat{a}_1$ wins with probability strictly less than one half.

1. Suppose $\hat{a}_1 < 2r_{\frac{1}{2}} - x$. Then, the second challenger may subsequently locate at $a_2 \in (\max\{r_{\frac{1}{2}}, \hat{a}_1\}, 2r_{\frac{1}{2}} - x)$ and win with probability one. This implies that for any best response of the second challenger after $\hat{a}_1 < 2r_{\frac{1}{2}} - x$, the first challenger loses with probability one, and thus $\hat{a}_1$ is not a best response.
2. Suppose $\hat{a}_1 > 2r_{\frac{1}{2}} - x$. If, after $\hat{a}_1$, the second challenger chooses $a_2 = out$, the first challenger loses with probability one. If, instead, the second challenger chooses $a_2 \in \mathbb{R}$, Claim 3 of Proposition 1, the supposition that the first and second challenger each win with positive probability by $\hat{a}_1, a_2 \in \mathbb{R}$ and $q > 0$ implies that no party wins in the first period with positive probability, i.e., there is a second round contest in which both challengers participate with positive probability. Then, $x < r_{\frac{1}{2}} < 2r_{\frac{1}{2}} - x < \hat{a}_1$ implies $a_2 \in \mathbb{R}$ is a best response only if $a_2 \in (2r_{\frac{1}{2}} - \hat{a}_1, \hat{a}_1)$, so that the party located at $\hat{a}_1$ loses with probability one.

□

This implies that if the established party that is supposed to choose *out* after the null history instead locates at x satisfying $r_{\frac{1}{2}} > x > \max\{r_p, r_{\frac{1}{2}-q}\}$, $2r_{\frac{1}{2}} - x < \min\{r_{1-p}, r_{\frac{1}{2}+q}\}$, and $F(2r_{\frac{1}{2}} - x) - F(x) + q < \min\{F(x), 1 - F(2r_{\frac{1}{2}} - x)\}$, it derives a payoff of one half on the equilibrium path of every subgame perfect equilibrium after the history h^1, and thus the deviation is profitable. We conclude that if $p < .5$ and $q \in (0, .25)$, there is no equilibrium in which two parties locate at the median voters' most preferred policy, i.e., Proposition 1 does not extend to the setting with two challengers.
What if, instead, $p < .5$ and $q = 0$? Example 3 showed that if the distribution of voter preferences is uniform on the unit interval, an equilibrium in which a single established party and a single challenger locate at the median voter's most preferred platform exists if and only if $p \leq .375$. I now extend Example 3 to a wider class of distribution functions, and $n \geq 2$ established parties.

Proposition A3. Suppose there are $n \geq 2$ established parties and two challengers, and consider a run-off rule with $q = 0$. For any distribution of voter ideal policies that admits a strictly quasi-concave density that is symmetric around $r_{\frac{1}{2}}$, there exists $p^* < .5$ such that if and only if $p \leq p^*$, an equilibrium profile exists in which a single established party and a single challenger locate at $r_{\frac{1}{2}}$; the other $n-1$ established parties stay *out*, and the other challenger stays *out*.

Proof. I begin by proving the *If* part. I first characterize the threshold p^*.
Step 1: Constructing p^*. Strict quasi-concavity and symmetry of the density of voter preferences implies that for any $x < r_{\frac{1}{2}}$:

$$r_{\frac{1}{2}} \in \arg \max_{y \in [x, 2r_{\frac{1}{2}} - x]} \{F(.5(2r_{\frac{1}{2}} - x + y)) - F(.5(x + y)\} \tag{21}$$

By Lemma 4.1 and Theorem 2.11 of Weber (1992), quasi-concavity of $f(\cdot)$ implies that there exists a unique policy $x^* \in (r_{\frac{1}{4}}, r_{\frac{1}{3}})$ such that $F(x^*) = F(.5(2r_{\frac{1}{2}} - x^* + r_{\frac{1}{2}})) - F(.5(x^* + r_{\frac{1}{2}})) = 1 - F(2r_{\frac{1}{2}} - x^*)$. Set $p^* = F(.5(x^* + r_{\frac{1}{2}}))$ and fix a run-off rule satisfying $q = 0$ and $p \leq p^*$.
Step 2: Strategies. I let $a_1 \in \mathbb{R} \cup \{out\}$ denote the action of the first challenger, and $a_2 \in \mathbb{R} \cup \{out\}$ denote the action of the second challenger. To support an equilibrium, I specify the following strategy: (i) at date 1, a single established party locates at the policy $r_{\frac{1}{2}}$, and the remaining established parties select *out*; (ii) after any history h^2, an equilibrium of the subgame is played. Existence of an equilibrium in the subgame beginning after any history $h^2 \in H^2$ is obtained by a straightforward extension of Lemma 2.
Step 3: Existence if $p \leq p^*$. Recall that the first challenger chooses an action after the history h^2. Familiar arguments from the setting with a single challenger establish that if $p < .5$ and $q = 0$, after a history h^2 with $X(h^2) = \{r_{\frac{1}{2}}\}$, $\varrho(r_{\frac{1}{2}}, h^2) = 1$, the actions $a_1 = out$, and $a_2 = r_{\frac{1}{2}}$ are chosen on the path of every subgame perfect equilibrium. Moreover, if an established party $i \in \{1, ..., n\}$ locates at $r_{\frac{1}{2}}$ after the null history and all other established parties choose *out*, the established party i wins with probability one half—it is straightforward to verify that any deviation by this established party to another location in $\mathbb{R}$ results in a probability of winning weakly less than one half in every equilibrium of the subgame after this deviation. Thus, to prove existence, it is sufficient to rule out a profitable deviation for one of the $n-1$ established parties that is supposed to choose *out*. To establish this, in turn, it is sufficient to show that if $p \leq p^*$, there exists an equilibrium of the subgame beginning after history $h^2 \in H^2$, with associated platforms $X(h^2)$, such that:

(a) if $X(h^2) = \{x_1, r_{\frac{1}{2}}\}$, $x_1 < r_{\frac{1}{2}}$, $\varrho(x_1, h^2) = \varrho(r_{\frac{1}{2}}, h^2) = 1$, the party located at x_1 loses with probability one,

(b) if $X(h^2) = \{r_{\frac{1}{2}}\}$, $\varrho(r_{\frac{1}{2}}, h^2) = 2$, the parties located at $r_{\frac{1}{2}}$ lose with probability one, and

(c) if $X(h^2) = \{r_{\frac{1}{2}}, x_2\}$, $x_2 > r_{\frac{1}{2}}$, $\varrho(r_{\frac{1}{2}}, h^2) = \varrho(x_2, h^2) = 1$, the party located at x_2 loses with probability one.

When these conditions are satisfied, there exists an equilibrium such that on the equilibrium path: after the null history, a single established party locates at $r_{\frac{1}{2}}$, and the remaining established parties select *out*, since after any history satisfying (a) through (c), we may specify that the corresponding equilibrium of the subgame is played. I focus on establishing existence of an equilibrium of the subgame beginning after history h^2 that satisfies (a); (b) is easy and (c) is a symmetric argument for case (a). I begin by proving two claims. Claim 1 establishes that condition (a) is satisfied if, after the history h^2 specified in (a), $a_1 = 2r_{\frac{1}{2}} - x_1$ is *not* played by the challenger in an equilibrium. Claim 2 establishes that either (1) $a_1 = 2r_{\frac{1}{2}} - x_1$ is not a best response for the challenger after the history specified in (a), or that (2) there exists an equilibrium after this history in which $a_1 = 2r_{\frac{1}{2}} - x_1$ and the established party located at x_1 loses with probability one.

Claim 1. In every equilibrium of the subgame beginning after a history h^2 such that $X(h^2) = \{x_1, r_{\frac{1}{2}}\}$, $x_1 < r_{\frac{1}{2}}$, $\varrho(x_1, h^2) = \varrho(r_{\frac{1}{2}}, h^2) = 1$, the established party located at x_1 loses with probability one if the first challenger chooses $a_1 \neq 2r_{\frac{1}{2}} - x_1$.

Proof. I consider each possible choice of action by the first challenger, $a_1 \in \mathbb{R} \cup \{out\}$, *excluding* the action $2r_{\frac{1}{2}} - x_1$, after the history h^2 with properties specified in the Claim. I show (a) if $a_1 = out$ is chosen, the established party located at x_1 loses and (b) for any other action $a_1 \in \mathbb{R} \setminus \{2r_{\frac{1}{2}} - x_1\}$, either (i) for any best response of the second challenger, the first challenger strictly prefers to select *out* rather than action $a_1 \in \mathbb{R} \setminus \{2r_{\frac{1}{2}} - x_1\}$, or (ii) whenever the second challenger plays a best response a_2 after the first challenger's action a_1, the established party located at x_1 loses with probability one.

1. If $a_1 = out$, the second challenger may locate at a policy $r_{\frac{1}{2}} + \epsilon$ for $\epsilon > 0$ satisfying $1 - F(r_{\frac{1}{2}} + \frac{1}{2}\epsilon) > \max\{p, F(r_{\frac{1}{2}} + \frac{1}{2}\epsilon) - F(.5(x_1 + r_{\frac{1}{2}})), F(.5(x_1 + r_{\frac{1}{2}}))\}$, thereby winning outright in the first round. This implies the second challenger wins with probability one whenever it plays a best response after the action $a_1 = out$.

2. If $a_1 \leq r_{\frac{1}{2}}$, the second challenger may locate at a policy $r_{\frac{1}{2}} + \epsilon$ for $\epsilon > 0$ satisfying $1 - F(r_{\frac{1}{2}} + \frac{1}{2}\epsilon) > \max\{p, F(r_{\frac{1}{2}} + \frac{1}{2}\epsilon) - F(.5(x_1 + r_{\frac{1}{2}})), F(.5(x_1 + r_{\frac{1}{2}}))\}$, thereby winning outright in the first round. This implies the second challenger wins with probability one after an action $a_1 \leq r_{\frac{1}{2}}$, and so $a_1 \leq r_{\frac{1}{2}}$ is not a best response.

3. Suppose $a_1 \in (r_{\frac{1}{2}}, 2r_{\frac{1}{2}} - x_1)$. I consider the set of possible actions $a_2 \in \mathbb{R} \cup \{out\}$ of the second challenger, after this history. I show (a) if $a_2 = out$ is chosen, the established party located at x_1 loses with probability one, and (b) for any other action $a_2 \in \mathbb{R}$, either (i) the first challenger would strictly prefer $a_1 = out$ if a_2 is played, or (ii) the second challenger strictly prefers $a_2 = out$ to $a_2 \in \mathbb{R}$, or (iii) the established party located at x_1 loses with probability one when a_2 is chosen, after $a_1 \in (r_{\frac{1}{2}}, 2r_{\frac{1}{2}} - x_1)$.

   3.a If $a_2 = out$, the established party located at x_1 wins a strictly lower share of the vote than the party located at $a_1 \in (r_{\frac{1}{2}}, 2r_{\frac{1}{2}} - x_1)$, and thus loses in a first or second round with probability one.

   3.b Suppose, instead, $a_2 \in \mathbb{R}$. If $\mathcal{X}(h^4)$ has the property that $\max_i v_i(\mathcal{X}(h^4)) \geq p$, a second round takes place, with probability zero; otherwise, a second round takes place, with probability one.

   3.b.i *Suppose that with probability one a second round takes place*, after a_2 is chosen. This implies that a_2 is a best response only if $a_2 \in [x_1, 2r_{\frac{1}{2}} - x_1)$, otherwise the second challenger loses in any second round contest with probability one.

   3.b.i.1 If $a_2 \in (x_1, 2r_{\frac{1}{2}} - x_1)$, the party located at x_1 loses in a second round contest against every other party, since $x_1 < \min\{a_2, a_1\} < \max\{a_2, a_1\} < 2r_{\frac{1}{2}} - x_1$.

   3.b.i.2 If $a_2 = x_1$, the second challenger's first-round vote share is $.5F(.5(x_1 + r_{\frac{1}{2}})) < .25$, since $x_1 < r_{\frac{1}{2}}$. This implies at least one party located at either $r_{\frac{1}{2}}$ or $a_1 \in (r_{\frac{1}{2}}, 2r_{\frac{1}{2}} - x_1)$ wins a vote share strictly higher than .25. This implies the second challenger loses with probability one in a second round, so $a_2 = out$ is strictly preferred to $a_2 = x_1$.

   3.b.ii *Suppose that with probability one a second round does not take place,* after a_2 is chosen.

   3.b.ii.1 If $a_2 < r_{\frac{1}{2}}$, then $\max\{a_2, x_1\} < r_{\frac{1}{2}}$. Since $\varrho(r_{\frac{1}{2}}, h^2) = 1$, either the party located at a_2 or the party located at x_1 wins a vote share strictly less than .25; since there are at most four parties positioned at platforms in $\mathbb{R}$ and vote shares must sum to one, at least one other party must win a vote weakly greater than .25. This implies that either the established party located at x_1 or the challenger located at a_2 loses with probability one in the first round.

3.b.ii.2 If $a_2 > r_{\frac{1}{2}}$, then $\min\{a_2, a_1\} > r_{\frac{1}{2}}$. This implies that either the party located at a_2 or the party located at a_1 wins a vote share strictly less than .25; as in the previous step, this implies that at least one of $a_1 \in \mathbb{R}$ or $a_2 \in \mathbb{R}$ is not a best response so that either $a_2 = out$ is strictly preferred by the second challenger, or $a_1 = out$ is strictly preferred by the first challenger.

3.b.ii.3 If $a_2 = r_{\frac{1}{2}}$, the party located at x_1 wins a vote share in the first round that is strictly lower than the vote share of the party located at $a_1 \in (r_{\frac{1}{2}}, 2r_{\frac{1}{2}} - x_1)$, by symmetry and quasi-concavity of $f(\cdot)$, so the established party located at x_1 loses with probability one in the first round.

4. Suppose $a_1 > 2r_{\frac{1}{2}} - x_1$. I consider the set of possible actions $a_2 \in \mathbb{R} \cup \{out\}$ of the second challenger, after this history. I show that for any best response by the second challenger $a_2 \in \mathbb{R} \cup \{out\}$, either (i) the first challenger would strictly prefer $a_1 = out$ if a_2 is played, i.e., $a_1 > 2r_{\frac{1}{2}} - x_1$ is not part of an equilibrium of the subgame after h^2, or (ii) the established party located at x_1 loses with probability one when a_2 is chosen after a_1.

4.a If $a_2 = out$, symmetry and quasi concavity of $f(\cdot)$ implies the first challenger wins a strict lower share of the vote than the established party located at $x_1 \in (2r_{\frac{1}{2}} - a_1, r_{\frac{1}{2}})$ and therefore loses with probability one in either a first or second round, so that a_1 is not a best response.

4.b If $a_2 < r_{\frac{1}{2}}$, at least one of the established party located at x_1 or the second challenger obtains a vote share strictly less than .25. If there is no second round after a_2 is chosen, either the established party located at x_1 loses with probability one in the first round, or the second challenger located at a_2 loses with probability one in the first round. If the former, the Claim follows. If the latter, the second challenger strictly prefers $a_2 = out$. Suppose, instead, there is a second round contest after a_2 is chosen. If the challenger located at a_1 proceeds to a second round with probability zero, the claim is true. Suppose, instead, the challenger located at a_1 proceeds to a second round with positive probability. Then, $a_2 < r_{\frac{1}{2}}$ is a best response only if $a_2 \in (2r_{\frac{1}{2}} - a_1, r_{\frac{1}{2}})$, so that the first challenger located at $a_1 > r_{\frac{1}{2}} > \min\{a_2, x_1\} > 2r_{\frac{1}{2}} - a_1$ loses the election with probability one in a second round. Thus, the first challenger strictly prefers the action *out* if the second challenger's best response is $a_2 < r_{\frac{1}{2}}$.

4.c If $a_2 = r_{\frac{1}{2}}$, quasi concavity and symmetry of $f(\cdot)$ implies that the first challenger located at $a_1 > 2r_{\frac{1}{2}} - x_1$ wins a strict lower first-round vote share than the party located at $x_1 \in (2r_{\frac{1}{2}} - a_1, r_{\frac{1}{2}})$ and therefore loses with probability one in either a first or second round. This implies that the challenger strictly prefers to choose *out* instead of $a_1 < 2r_{\frac{1}{2}} - x_1$.

4.d Suppose $a_2 > r_{\frac{1}{2}}$. This implies that either the first challenger located at a_1 or the second challenger located at a_2 obtains a vote share strictly less than .25. If there is no second round, this implies at least one of these parties loses with probability one. If there is a second round contest, then the challenger located at $a_2 > r_{\frac{1}{2}}$ wins with positive probability only if $a_2 \leq a_1$. If $a_2 \in (r_{\frac{1}{2}}, a_1)$, the first challenger loses in any second round contest, since $a_1 > a_2 > r_{\frac{1}{2}} > x_1 > 2r_{\frac{1}{2}} - a_1$. If $a_2 = a_1$, each receives a vote share strictly less than the established party located at $x_1 \in (2r_{\frac{1}{2}} - a_1, r_{\frac{1}{2}})$, and thus each challenger loses in any second round contest.

This completes the proof of the first Claim. $\square$

Claim 2. Let $p \leq p^*$. In an equilibrium of the subgame after a history h^2 such that $X(h^2) = \{x_1, r_{\frac{1}{2}}\}$, $x_1 < r_{\frac{1}{2}}$, $\varrho(x_1, h^2) = \varrho(r_{\frac{1}{2}}, h^2) = 1$, either (a) an equilibrium exists in which the first challenger plays either $a_1 \in out$, or $a_1 \in \mathbb{R} \setminus \{2r_{\frac{1}{2}} - x_1\}$, and the established party located at x_1 loses the election with probability one, or (b) an equilibrium exists in which $a_1 = 2r_{\frac{1}{2}} - x_1$ is chosen, and the established party located at x_1 loses the election with probability one.

Proof. I index the possible cases according to the set of locations $x_1 < r_{\frac{1}{2}}$.

1. Suppose $x_1 < 2r_{\frac{1}{4}} - r_{\frac{1}{2}}$. If $a_1 = 2r_{\frac{1}{2}} - x_1$, $a_2 = r_{\frac{1}{2}}$ is a strict best response, and the first challenger loses with probability one.

2. Suppose $x_1 \in [2r_{\frac{1}{4}} - r_{\frac{1}{2}}, x^*)$. Let $\bar{x}$ denote the platform that solves $1 - F(\bar{x}(x_1)) = F(.5(r_{\frac{1}{2}} + \bar{x}(x_1))) - F(.5(r_{\frac{1}{2}} + x_1))$, which exists by continuity of $F(\cdot)$. Since x^* solves $F(x^*) = 1 - F(2r_{\frac{1}{2}} - x^*) = F(.5(r_{\frac{1}{2}} + 2r_{\frac{1}{2}} - x^*)) - F(.5(x^* + r_{\frac{1}{2}}))$ and $x_1 < x^*$, we conclude $\bar{x}(x_1) < 2r_{\frac{1}{2}} - x^* < 2r_{\frac{1}{2}} - x_1$, and therefore $1 - F(.5(r_{\frac{1}{2}} + \bar{x}(x_1))) > 1 - F(.5(r_{\frac{1}{2}} + 2r_{\frac{1}{2}} - x^*)) = F(.5(x^* + r_{\frac{1}{2}})) \geq p$.

   Let the first challenger locate at $\bar{x}(x_1)$. I claim that, after this history, the unique best response of the second challenger is $a_2 = out$, and thus the first challenger wins with probability one. The action a_1 is therefore shown to be a best response for the challenger. To establish this claim, I show that $a_2 \in \mathbb{R}$ after this history implies the second challenger loses the election with probability one.

   2.a Suppose $a_2 \leq x_1$. Since $F(.5(a_2 + x_1)) \leq F(x_1) = 1 - F(x_1) < 1 - F(\bar{x}(x_1)) = F(.5(r_{\frac{1}{2}} + \bar{x}(x_1))) - F(.5(x_1 + r_{\frac{1}{2}})) < 1 - F(.5(r_{\frac{1}{2}} + \bar{x}(x_1)))$, the second challenger loses with probability one in the first round.

   2.b Suppose $a_2 \in (x_1, r_{\frac{1}{2}})$. Since $F(\cdot)$ admits a strictly quasi-concave and symmetric density, $F(z)$ is convex in $z \leq r_{\frac{1}{2}}$. Thus:

$$F(.5(a_2 + r_{\frac{1}{2}})) - F(.5(x_1 + a_2)) \leq .5 - F(.5(x_1 + r_{\frac{1}{2}})) \tag{22}$$

$$< .5 - F(.5(x_1 + r_{\frac{1}{2}})) + F(.5(r_{\frac{1}{2}} + \bar{x}(x_1))) - .5$$

$$= 1 - F(\bar{x}(x_1)))$$

$$< 1 - F(.5(r_{\frac{1}{2}} + \bar{x}(x_1))), \tag{23}$$

   where the last line is the vote share of the first challenger, located at $a_1 = \bar{x}(x_1)$. Combined with $1 - F(.5(r_{\frac{1}{2}} + \bar{x}(x_1))) > p$, we conclude that the second challenger loses in the first round if she locates at a_2.

   2.c Suppose $a_2 = r_{\frac{1}{2}}$. Since $.5(F(.5(r_{\frac{1}{2}} + \bar{x}(x_1))) - F(.5(x_1 + r_{\frac{1}{2}}))) < F(.5(r_{\frac{1}{2}} + \bar{x}(x_1))) - F(.5(x_1 + r_{\frac{1}{2}})) = 1 - F(\bar{x}(x_1)) < 1 - F(.5(r_{\frac{1}{2}} + \bar{x}(x_1)))$, and $1 - F(.5(r_{\frac{1}{2}} + \bar{x}(x_1))) > p$, the second challenger loses in the first round if she locates at a_2.

   2.d Suppose $a_2 \in (r_{\frac{1}{2}}, \bar{x}(x_1))$. $F(\cdot)$ unimodal and symmetric implies $F(z)$ is weakly concave in $z \geq r_{\frac{1}{2}}$. Thus:

$$F(.5(a_2 + \bar{x}(x_1))) - F(.5(r_{\frac{1}{2}} + a_2)) \leq F(.5(r_{\frac{1}{2}} + \bar{x}(x_1))) - .5 \tag{24}$$

$$< .5 - F(.5(x_1 + r_{\frac{1}{2}})), \tag{25}$$

where the second inequality arises from the fact that $\bar{x}(x_1) < 2r_{\frac{1}{2}} - x_1$, which—using symmetry of the density $f(\cdot)$—implies:

$$F(.5(x_1 + r_{\frac{1}{2}})) + F(.5(r_{\frac{1}{2}} + \bar{x}(x_1))) < 1 - F(.5(r_{\frac{1}{2}} + \bar{x}(x_1))) + F(.5(r_{\frac{1}{2}} + \bar{x}(x_1))) = 1, \tag{26}$$

and combining the outer expressions yields the second inequality in (25). So, $a_2 \in (r_{\frac{1}{2}}, \bar{x}(x_1))$ implies the second challenger wins a vote share strictly less than that won by the party located at $r_{\frac{1}{2}}$. This implies that the second challenger loses either in the first or the second round.

2.e Suppose $a_2 = \bar{x}(x_1)$. I show that the second challenger loses with probability one to the established party located at $r_{\frac{1}{2}}$, by proving:

$$.5(1 - F(.5(r_{\frac{1}{2}} + \bar{x}(x_1)))) < F(.5(r_{\frac{1}{2}} + \bar{x}(x_1))) - F(.5(r_{\frac{1}{2}} + x_1)). \tag{27}$$

To establish this inequality, I use again the assumption that $f(\cdot)$ is strictly quasi-concave and symmetric and the fact that since $.5(r_{\frac{1}{2}} + a_2) > r_{\frac{1}{2}}$ for $a_2 > r_{\frac{1}{2}}$, we have:

$$\begin{aligned} F(\bar{x}(x_1)) - F(.5(r_{\frac{1}{2}} + \bar{x}(x_1))) &\leq F(.5(r_{\frac{1}{2}} + \bar{x}(x_1))) - .5 \\ &< F(.5(r_{\frac{1}{2}} + \bar{x}(x_1))) - .5 + .5 - F(.5(r_{\frac{1}{2}} + x_1)). \end{aligned} \tag{28}$$

Moreover, by definition of $\bar{x}(x_1)$:

$$1 - F(\bar{x}(x_1)) = F(.5(r_{\frac{1}{2}} + \bar{x}(x_1))) - F(.5(r_{\frac{1}{2}} + x_1)). \tag{29}$$

Combining expression (28) and (29), we obtain:

$$\begin{aligned} .5(1 - F(.5(r_{\frac{1}{2}} + \bar{x}(x_1))) &= .5(1 - F(\bar{x}(x_1)) + F(\bar{x}(x_1)) - F(.5(r_{\frac{1}{2}} + \bar{x}(x_1)))) \\ &< F(.5(r_{\frac{1}{2}} + \bar{x}(x_1))) - F(.5(r_{\frac{1}{2}} + x_1))), \end{aligned} \tag{30}$$

i.e., inequality (27) holds. This implies that if $a_2 = a_1 = \bar{x}(x_1)$, the second challenger wins a strictly lower vote share than the established party located at $r_{\frac{1}{2}}$, and therefore loses in either a first or second round.

2.f Suppose $a_2 > \bar{x}(x_1)$. Since:

$$1 - F(.5(\bar{x}(x_1) + a_2)) < 1 - F(\bar{x}(x_1)) = (F(.5(r_{\frac{1}{2}} + \bar{x}(x_1))) - F(.5(x_1 + r_{\frac{1}{2}})), \tag{31}$$

the second challenger wins a strictly lower vote share than the established party located at $r_{\frac{1}{2}}$, and therefore loses either in a first or second round.

3. Suppose $x_1 \in [x^*, r_{\frac{1}{2}})$. Then, $F(.5(x_1 + r_{\frac{1}{2}})) \geq p$, and $F(.5(r_{\frac{1}{2}} + 2r_{\frac{1}{2}} - x_1)) - F(.5(x_1 + r_{\frac{1}{2}})) \leq F(x_1) = 1 - F(2r_{\frac{1}{2}} - x_1)$. Let the first challenger locate at a platform $a_1 = 2r_{\frac{1}{2}} - x_1 - \epsilon$ for $\epsilon \in (0, r_{\frac{1}{2}} - x_1)$ satisfying $1 - F(2r_{\frac{1}{2}} - x_1 - \epsilon) < F(.5(x_1 + r_{\frac{1}{2}}))$. After this history, the second challenger strictly prefers the action $a_2 = out$, and the first challenger located at a_1 wins the election in the first round.

This completes the proof of the second Claim. □

These steps establish that, after a history h^2 such that $X(h^2) = \{x_1, r_{\frac{1}{2}}\}$, for any $x_1 < r_{\frac{1}{2}}$, $\varrho(x_1) = \varrho(r_{\frac{1}{2}}) = 1$, an equilibrium exists in which the party located at x_1 loses with probability one. Together with parts (b) and (c), we conclude that if $p \leq p^*$ and $q = 0$ an equilibrium profile exists in which one established party locates at $r_{\frac{1}{2}}$, the $n-1$ remaining established parties each choose *out*, the first challenger chooses *out*, and the second challenger locates at $r_{\frac{1}{2}}$.

Step 4: Only if $p \leq p^*$. Suppose $p > F(.5(x^* + r_{\frac{1}{2}}))$ and $q = 0$. Consider any strategy profile specifying that, after the null history, a single established party locates at $r_{\frac{1}{2}}$, and the remaining $n-1$ established parties choose *out*. Let one of these latter $n-1$ established parties instead locate at $x^* < r_{\frac{1}{2}}$, solving $F(x^*) = F(.5(2r_{\frac{1}{2}} - x^* + r_{\frac{1}{2}})) - F(.5(x^* + r_{\frac{1}{2}})) = 1 - F(2r_{\frac{1}{2}} - x^*)$. I claim that on the path of every subgame perfect equilibrium beginning after a history h^2 satisfying $X(h^2) = \{x^*, r_{\frac{1}{2}}\}$, $\varrho(x_1, h^2) = \varrho(x_2, h^2) = 1$, the first challenger locates at $a_1 = 2r_{\frac{1}{2}} - x^*$, the second challenger chooses $a_2 = out$, and the established party located at x^* wins the election with probability .5.

Claim 3. After a history h^3 satisfying $X(h^3) = \{x^*, r_{\frac{1}{2}}, 2r_{\frac{1}{2}} - x^*\}$, and $\varrho(x^*, h^3) = \varrho(r_{\frac{1}{2}}, h^3) = \varrho(2r_{\frac{1}{2}} - x^*, h^3) = 1$, the second challenger's unique best response is $a_2 = out$.

Proof. If $a_2 \leq x_1$, the second challenger wins a strictly lower share of the vote than the established party located at $r_{\frac{1}{2}}$ and the challenger located at $2r_{\frac{1}{2}} - x^*$. If $a_2 \geq 2r_{\frac{1}{2}} - x^*$, the second challenger wins a strictly lower share of the vote than the established party located at $r_{\frac{1}{2}}$ and the established party located at x^*. If the challenger locates at $a_2 \in (x^*, 2r_{\frac{1}{2}} - x^*)$, she wins a strictly lower share of the vote than the two parties located at x^* and $2r_{\frac{1}{2}} - x^*$. □

So, it is sufficient to show that, after history h^2 such that $X(h^2) = \{x^*, r_{\frac{1}{2}}\}$, $\varrho(x_1, h^2) = \varrho(x_2, h^2) = 1$, the first challenger strictly prefers the action $2r_{\frac{1}{2}} - x^*$, since this implies that the established party located at x^* wins with probability .5. I consider an alternative location choice for the first challenger, $a_1 \in \mathbb{R} \setminus \{2r_{\frac{1}{2}} - x^*\}$. Notice that $a_1 = out$ cannot be a best response, since by locating at $a_1 = 2r_{\frac{1}{2}} - x^*$, the previous Claim implies the first challenger wins the election with probability one half.

1. Suppose $a_1 \leq r_{\frac{1}{2}}$. The second challenger may locate at $a_2 = r_{\frac{1}{2}} + \epsilon$ for $\epsilon > 0$ satisfying $1 - F(r_{\frac{1}{2}} + \frac{1}{2}\epsilon) > \max\{p, F(r_{\frac{1}{2}} + \frac{1}{2}\epsilon) - F(.5(x^* + r_{\frac{1}{2}})), F(.5(x^* + r_{\frac{1}{2}}))\}$, winning with probability one in the first round. So, the second challenger always wins with probability one whenever it plays a best response after $a_1 \leq r_{\frac{1}{2}}$ is chosen by the first challenger, and so the first challenger strictly prefers $a_1 = 2r_{\frac{1}{2}} - x^*$.

2. Suppose $a_1 \in (r_{\frac{1}{2}}, 2r_{\frac{1}{2}} - x^*)$. The second challenger may locate at $a_2 \in (a_1, 2r_{\frac{1}{2}} - x^*)$; since the second challenger and the party located at x_1 win the strictly highest vote share in the first round, and since $F(.5(x^* + r_{\frac{1}{2}})) < p$, either (a) the second challenger wins in the first round, or (b) the second challenger and the party located at x_1 participate in a second round, in which the second challenger wins with probability one. This implies that the second challenger must win with probability one whenever it plays a best response after $a_1 \in (r_{\frac{1}{2}}, 2r_{\frac{1}{2}} - x^*)$, and so the first challenger strictly prefers $a_1 = 2r_{\frac{1}{2}} - x^*$.

3. Suppose $a_1 > 2r_{\frac{1}{2}} - x^*$. If $a_2 = out$ is a best response, the first challenger loses either in the first or second round, and so the first challenger strictly prefers $a_1 = 2r_{\frac{1}{2}} - x^*$. If, instead, $a_2 \in \mathbb{R}$ is a best response, either (1) with probability one, there is a second round contest or (2) with probability one, there is no second round contest.

   3.a *Suppose there is no second round contest* after $a_2 \in \mathbb{R}$ is chosen.

   3.a.1 $a_2 \leq x^*$ yields a strictly lower vote share for the second challenger than the party located at $r_{\frac{1}{2}}$, since $F(.5(a_2 + x^*)) < F(x^*) = F(.5(r_{\frac{1}{2}} + 2r_{\frac{1}{2}} - x^*)) - F(.5(x^* + r_{\frac{1}{2}})) < F(.5(r_{\frac{1}{2}} + a_1)) - F(.5(x^* + r_{\frac{1}{2}}))$, so $a_2 \leq x^*$ is not a best response.

3.a.2 $a_2 \in (x^*, r_{\frac{1}{2}})$ yields a strictly lower vote share for the second challenger than the party located at x^*, by construction of x^*. I conclude that $a_2 \in (x^*, r_{\frac{1}{2}})$ is not a best response.

3.a.3 If $a_2 = r_{\frac{1}{2}}$, the first challenger located at $a_1 > 2r_{\frac{1}{2}} - x^*$ wins a strictly lower share of the vote than the established party located at x^*, by symmetry of the distribution of voter ideal points.

3.a.4 If $a_2 > r_{\frac{1}{2}}$, $\varrho(r_{\frac{1}{2}}) = 1$ and $\min\{a_1, a_2\} > r_{\frac{1}{2}}$ implies $1 - F(.5(\min\{a_1, a_2\} + r_{\frac{1}{2}})) < .5$ and therefore either the second challenger or the first challenger receives a vote share strictly less than .25. This implies that either the party located at $r_{\frac{1}{2}}$ or the party located at x^* wins a strictly higher first-round vote share. Since there is no second round contest, by supposition, this implies that at least one challenger loses with probability one. This implies that either a_1 or a_2 is not a best response.

3.b *Suppose there is a second round contest.* Since $a_1 - r_{\frac{1}{2}} > r_{\frac{1}{2}} - x_1$, $a_2 \in \mathbb{R}$ is therefore a best response only if $a_2 \in (2r_{\frac{1}{2}} - a_1, a_1]$. If $a_2 \in (2r_{\frac{1}{2}} - a_1, a_1)$, the first challenger loses in a second round contest, so $a_1 = 2r_{\frac{1}{2}} - x^*$ is strictly preferred to a_1. If, instead, $a_2 = a_1$, $F(.5(r_{\frac{1}{2}} + a_1)) > .5$ implies $.5(1 - F(.5(r_{\frac{1}{2}} + a_1))) < .25$, so either the party located at $r_{\frac{1}{2}}$ or the party located at $x^* > 2r_{\frac{1}{2}} - a_1$ wins a strictly higher share of the vote. Since $a_1 > r_{\frac{1}{2}} > x_1 > 2r_{\frac{1}{2}} - a_1$, this implies the two challengers lose with probability one in any second round contest. We conclude that a_1 is not a best response.

I have shown that for any $a_1 \neq 2r_{\frac{1}{2}} - x^*$, in an equilibrium of the subgame beginning after the history in which a_1 is played, the first challenger wins with probability strictly less than one half. We conclude that after a history h^2 satisfying $X(h^2) = \{x^*, r_{\frac{1}{2}}\}$, $\varrho(x^*, h^2) = \varrho(r_{\frac{1}{2}}, h^2) = 1$, there is a unique subgame perfect equilibrium path of play, in which the first challenger selects $a_1 = 2r_{\frac{1}{2}} - x^*$, and the second challenger chooses *out*. Thus, there is no equilibrium in which, after the null history, a single established party locates at $r_{\frac{1}{2}}$ and the remaining established parties each select *out*, since a single established party that is supposed to choose *out* strictly prefers to locate at x^*. This concludes the *Only If* party of the Proposition. □

Supplemental Appendix D: Comparison with a Plurality Rule.

In this Supplemental Appendix, I extend Example 4, in the main text. Recall that we are considering a setting with a single challenger.

Proposition A4.

1. Suppose the distribution of voter preferences satisfies the following property: for any $t \in (0, r_{\frac{1}{2}} - r_{\frac{1}{4}})$, $F(r_{\frac{1}{4}} + t) > F(r_{\frac{1}{2}} + t) - F(r_{\frac{1}{4}} + t)$. Let there be three established parties, i.e., $n = 3$. Under a plurality rule, an equilibrium profile $\mathcal{X} = (X, \varrho, a_c)$ exists in which $|X| = 2$, $x_1 = r_{\frac{1}{4}}$, $x_1 = 2r_{\frac{1}{2}} - x_1$, $\varrho(x_1) = \varrho(x_2) = 2$, and the challenger is positioned at $a_c = r_{\frac{1}{4}}$.

2. Suppose the distribution of voter preferences satisfies the following property: for any $t \in (0, r_{\frac{3}{4}} - r_{\frac{1}{2}})$: $1 - F(r_{\frac{1}{2}} + t) > F(r_{\frac{1}{2}} + t) - F(2r_{\frac{1}{2}} - r_{\frac{3}{4}} + t)$. Let there be three established parties, i.e., $n = 3$. Under a plurality rule, an equilibrium profile $\mathcal{X} = (X, \varrho, a_c)$ exists in which $|X| = 2$, $x_1 = 2r_{\frac{1}{2}} - r_{\frac{3}{4}}$, $x_2 = r_{\frac{3}{4}}$, $\varrho(x_1, h^2) = \varrho(x_2, h^2) = 2$, and the challenger is positioned at $a_c = r_{\frac{3}{4}}$.

The conditions hold, for example, for the uniform and normal distributions, implicitly requiring that the distribution not have too large a 'spike' around the median. Note that symmetry of the density is not required. The condition in part 1. of the Proposition ensures that, when two established parties are located at $2r_{\frac{1}{2}} - r_{\frac{1}{4}}$ and one established party is located at $r_{\frac{1}{4}}$, if the challenger locates at $a_c = r_{\frac{1}{4}} + 2t$ for any $t \in (0, r_{\frac{1}{2}} - r_{\frac{1}{4}})$—i.e., in between the platforms $r_{\frac{1}{4}}$ and $2r_{\frac{1}{2}} - r_{\frac{3}{4}}$—its vote share $F(r_{\frac{1}{2}} + t) - F(r_{\frac{1}{4}} + t)$ is strictly less than that of the established party located at $r_{\frac{1}{4}}$, $F(r_{\frac{1}{4}} + t)$. Under a plurality rule, this implies that the challenger loses the election with probability one.

Proof. I prove part 1, since part 2 is a straightforward extension. Suppose the distribution of voter preferences satisfies the following property: for any $t \in (0, r_{\frac{1}{2}} - r_{\frac{1}{4}})$, $F(r_{\frac{1}{4}} + t) > F(r_{\frac{1}{2}} + t) - F(r_{\frac{1}{4}} + t)$. I specify the following strategy. (i) After the null history, one established party locates at $x_1 = r_{\frac{1}{4}}$ and two established parties locate at $x_2 = 2r_{\frac{1}{2}} - r_{\frac{1}{4}}$. (ii) After any history h^2, the challenger plays a best response.

Claim 1. Suppose the distribution of voter ideal policies satisfies the following property: for any $t \in (0, r_{\frac{1}{2}} - r_{\frac{1}{4}})$, $F(r_{\frac{1}{4}} + t) > F(r_{\frac{1}{2}} + t) - F(r_{\frac{1}{4}} + t)$. After a history h^2 with $X(h^2) = \{x_1, x_2\}$, $x_1 = r_{\frac{1}{4}}$, $x_2 = 2r_{\frac{1}{2}} - r_{\frac{1}{4}}$, $\varrho(x_1, h^2) = 1$, $\varrho(x_2, h^2) = 2$, $a_c = x_1$ is a best response.

Proof. If the challenger locates at $a_c = x_1 = r_{\frac{1}{4}}$, it wins with probability .25. If the challenger locates at $a_c < x_1$, it wins a vote share $F(.5(a_c + r_{\frac{1}{4}})) < .25$, while the established party located at $x_1 = r_{\frac{1}{4}}$ wins a vote share $.5 - F(.5(a_c + r_{\frac{1}{4}})) = .5 - .25 + (.25 - F(.5(a_c + r_{\frac{1}{4}}))) > .25$, so the challenger loses. If the challenger locates at $a_c \in (x_1, x_2)$, the assumption that for any $t \in (0, r_{\frac{1}{2}} - r_{\frac{1}{4}})$, $F(r_{\frac{1}{4}} + t) > F(r_{\frac{1}{2}} + t) - F(r_{\frac{1}{4}} + t)$ implies that the established party located at x_1 wins with probability one. If the challenger locates at $a_c = x_2$, the established party located at $x_1 = r_{\frac{1}{4}}$ wins a vote share of .5, while the three parties located at x_2 win a vote share $\frac{1}{6}$, and therefore lose with probability one. If $a_c > 2r_{\frac{1}{2}} - r_{\frac{1}{4}}$, the challenger wins a vote share strictly less than $.5 = F(.5(x_1 + x_2))$, and therefore loses to the established party located at $x_1 = 2r_{\frac{1}{2}} - x_2$. □

I next show that there is no profitable deviation for any established party from the strategy specified in part (i).

1. If the established party that is supposed to locate at $x_1 = r_{\frac{1}{4}}$, under the strategy profile instead locates at $x' < r_{\frac{1}{4}}$ or $x' > r_{\frac{1}{4}}$, I specify that the challenger locates at $r_{\frac{1}{4}}$, which is a best response since she wins with probability one.

2. Consider, instead, an alternative location for one of the established parties that is supposed to locate at x_2, under the strategy profile.

   2.a. Suppose $x_2' < x_1$. For any action $a_c \in \mathbb{R} \cup \{out\}$, the party located at $x_2' < r_{\frac{1}{4}}$ wins a vote share $F(.5(x_2' + r_{\frac{1}{4}})) < .25$. But since there are four or fewer parties located at platforms after the action $a_c \in \mathbb{R} \cup \{out\}$ is chosen, at least one party wins a vote share which is weakly in excess of .25. Thus, the party located at x_2' loses the contest.

   2.b Suppose $x_2' > x_2$. If $x_2 \geq r_{\frac{3}{4}}$, a similar argument to 2.a can be applied. Suppose, instead, $x_2 < r_{\frac{3}{4}}$. If the challenge subsequently prefers $a_c = out$, the deviation is unprofitable, since the established party located at x_1 wins a vote share .5, while the deviating established party wins a vote share strictly less than .5. Suppose, instead, $a_c \in \mathbb{R}$ is a best response after this deviation to $x_2' > x_2$. We have that (1) $a_c \in \mathbb{R}$ is a best response only if $a_c = x_1$, since any other location yields the challenger a strictly lower share of the vote than the established party located at x_1, and (2) the parties located at $a_c = x_1$ and x_2' win with positive probability only if $1 - F(.5(x_2 + x_2')) = .25$. But this implies $F(.5(x_2 + x_2')) - F(.5(x_1 + x_2)) = .75 - .5 = .25$, so each party wins with probability .25, and thus the deviation to x_2' is not strictly profitable.

2.c Suppose $x_2' \in (x_1, x_2)$. I consider the set of all possible subsequent actions by the challenger, after this deviation.

2.c.1 Suppose $a_c = out$. Then, the deviation to x_2' is not profitable, since the party located at x_2' wins a strictly lower share of the vote than at least one party located either at $x_1 = r_{\frac{1}{4}}$ or $x_2 = 2r_{\frac{1}{2}} - r_{\frac{1}{4}}$.

2.c.2 Suppose $a_c \in [x_1, x_2]$. If $a_c = x_1$, each of the challenger and the established party located at x_1 wins a vote share strictly less than .25; since there are only four parties occupying platforms, this implies at least one other party wins a vote share weakly greater than .25. This implies $a_c = x_1$ is not a best response. If $a_c \in (x_1, x_2]$, the challenger wins a strictly lower share of the vote than the established party located at x_1. I conclude that $a_c \in [x_1, x_2]$ is strictly worse for the challenger than $a_c = out$.

2.c.3 If $a_c < x_1 = r_{\frac{1}{4}}$, the challenger wins a vote share $F(.5(a_c + r_{\frac{1}{4}})) = .25$; by similar reasoning to 2.c.2, the challenger therefore loses the election with probability one.

2.c.4 If $a_c > x_2$, the party located at x_2' wins a strictly lower vote share than the established party located at x_1, and therefore loses the election with probability one.

We conclude that $x_2' \in (x_1, x_2)$ is not a profitable deviation, for any best response of the challenger.

3. Suppose, finally, $x_2' = x_1$. After this action by the established party, if $a_c = out$, the established party located at $x_2' = x_1$ loses with probability one. Suppose, instead, $a_c \in \mathbb{R}$ after this deviation. This action is a best response only if $a_c > x_1$. If $a_c \in (x_1, x_2)$ is a best response, the established party located at x_2' wins a vote share strictly less than .25, while at least one other party wins a vote share weakly in excess of .25. If $a_c = x_2$ is a best response, each party wins the election with probability .25, so the deviation to $x_2' = x_1$ is not profitable. Suppose, finally, $a_c > x_2$ is a best response. The challenger wins with positive probability after choosing this action only if $1 - F(.5(x_2 + a_c) \geq \max\{.25, F(.5(x_2 + a_c) - .5\}$, which in turn implies $x_2 < r_{.75}$. If $1 - F(.5(x_2 + a_c) > .25$, the established party located at x_2' loses with probability one, since its vote share is .25. If $1 - F(.5(x_2 + a_c) = .25$, the challenger may locate instead at $a_c - \epsilon$ for $\epsilon \in (0, a_c - x_2)$, which implies $1 - F(.5(x_2 + a_c - \epsilon)) > \max\{.25, F(.5(x_2 + a_c) - .5\}$, thereby winning with probability one in the first round. This implies that the deviation to x_2' is not profitable.

This completes the argument. ☐

Supplemental Appendix E: Moving Thresholds. I formally define a 'moving threshold'.[9] The electoral rule is summarized by (p_H, p_L, q) where $p_L < p_H < \frac{1}{2}$ and $0 < q \leq .5 - p_L$. Let $W(\mathcal{X}; p_H, p_L, q)$ denote the set of parties such that *either* (1) $v_i(\mathcal{X}) \geq p_H$, *or* (2) $v_i(\mathcal{X}) \geq p_L$ and $v_i(\mathcal{X}) - \max_{j \neq i} v_j(\mathcal{X}) \geq q$. If $W(\mathcal{X}; p_H, p_L, q) \neq \emptyset$, each party contained in $W(\mathcal{X}; p_H, p_L, q)$ wins in the first round with probability $\frac{1}{|W(\mathcal{X}; p_H, p_L, q)|}$, where $|W(\mathcal{X}; p_H, p_L, q)|$ is the number of parties contained in $W(\mathcal{X}; p_H, p_L, q)$. Otherwise, the winner is the party which obtains a majority in the second round in a contest between the two parties which obtained the most votes in the first round. All ties in both rounds are resolved fairly, as in the benchmark presentation. A *moving threshold* is a triple (p_H, p_L, q) satisfying $p_L < p_H < .5$ and $0 < q \leq .5 - p_L$. For example, Argentina's moving threshold is $p_H = .45$, $p_L = .4$ and $q = .1$ and Nicaragua's moving threshold is $p_H = .4$, $p_L = .35$ and $q = .05$.

[9] This nomenclature was first proposed in Bouton (2013).

Proposition A5. Suppose there are $n \geq 2$ established parties, and a single challenger. For any moving threshold, an equilibrium exists in which, on the equilibrium path, a single established party locates at $r_{\frac{1}{2}}$, the remaining established parties stay out, and the challenger locates at $r_{\frac{1}{2}}$. Moreover, there is no equilibrium profile $\mathcal{X}$ satisfying $\sum_{x \in X} \varrho(x) \leq 2$ and in which $a_c = out$.

Proof. I specify the following strategy. (i) After the null history, one established party locates at $r_{\frac{1}{2}}$, and the remaining $n-1$ established parties choose *out*. (ii) If $X(h^2) = \{r_{\frac{1}{2}}\}$, $\varrho(r_{\frac{1}{2}}, h^2) = 1$, $a_c = r_{\frac{1}{2}}$. (iii) If $X(h^2) = \{x_1, r_{\frac{1}{2}}\}$, $x_1 < r_{\frac{1}{2}}$, $\varrho(x_1, h^2) = 1$, $\varrho(r_{\frac{1}{2}}, h^2) = 1$, $a_c = r_{\frac{1}{2}} + \epsilon$ where $\epsilon > 0$ satisfies $1 - F(r_{\frac{1}{2}} + \frac{1}{2}\epsilon) > \max\{p_H, F(.5(x_1 + r_{\frac{1}{2}})), F(r_{\frac{1}{2}} + \frac{1}{2}\epsilon) - F(.5(x_1 + r_{\frac{1}{2}}))\}$. (iv) If $X(h^2) = \{r_{\frac{1}{2}}, x_2\}$, $\varrho(r_{\frac{1}{2}}, h^2) = 1$, $\varrho(x_2, h^2) = 1$, $x_2 > r_{\frac{1}{2}}$, $a_c = r_{\frac{1}{2}} - \epsilon$ where $\epsilon > 0$ satisfies $F(r_{\frac{1}{2}} - \frac{1}{2}\epsilon) > \max\{p_H, 1 - F(.5(r_{\frac{1}{2}} + x_2)), F(.5(r_{\frac{1}{2}} + x_2)) - F(r_{\frac{1}{2}} - \frac{1}{2}\epsilon)\}$. (iv) If $X(h^2) = \{r_{\frac{1}{2}}\}$, $\varrho(r_{\frac{1}{2}}, h^2) = 2$, $a_c = r_{\frac{1}{2}} + \epsilon$ for $\epsilon > 0$ satisfying $1 - F(r_{\frac{1}{2}} + \frac{1}{2}\epsilon) > \max\{.5F(r_{\frac{1}{2}} + \frac{1}{2}\epsilon), p_H\}$. (v) After any other history, the challenger plays a best response. It is easy to verify that this is an equilibrium by way of similar arguments to the proof of Proposition 1. I next verify that there is no equilibrium profile $\mathcal{X} = (X, \varrho, a_c)$ satisfying $\sum_{x \in X} \varrho(x) \leq 2$, in which $a_c = out$. *Claim 1.* There is no equilibrium profile in which $X = \{x_1\}$, $\varrho(x_1) \in \{1, 2\}$, and $a_c = out$. *Proof.* If $x_1 \neq r_{\frac{1}{2}}$, or if $x_1 = r_{\frac{1}{2}}$ and $\varrho(x_1, h^2) = 1$, the challenger may locate at $r_{\frac{1}{2}}$ and win with positive probability. If $x_1 = r_{\frac{1}{2}}$ and $\varrho(x_1, h^2) = 2$, the challenger may locate at $a_c = r_{\frac{1}{2}} + \epsilon$ for $\epsilon > 0$ satisfying $1 - F(r_{\frac{1}{2}} + \frac{1}{2}\epsilon) > \max\{.5F(r_{\frac{1}{2}} + \frac{1}{2}\epsilon), p_H\}$, and win with probability one. □ *Claim 2.* There is no equilibrium with $X = \{x_1, x_2\}$, $\varrho(x_1) = \varrho(x_2) = 1$, $a_c = out$. *Proof.* In such an equilibrium, we must have $x_2 > r_{\frac{1}{2}} > x_1$, and $x_2 - r_{\frac{1}{2}} = r_{\frac{1}{2}} - x_1$. For any $t \in (0, r_{\frac{1}{2}} - x_1)$, the challenger the action *out* to the action $x_1 + 2t$ if and only either (1) $F(r_{\frac{1}{2}} + t) - F(x_1 + t) < \min\{F(x_1 + t), 1 - F(r_{\frac{1}{2}} + t)\}$, or (2) $F(r_{\frac{1}{2}} + t) - F(x_1 + t) \geq \min\{F(x_1 + t), 1 - F(r_{\frac{1}{2}} + t)\}$ and either (2i) $\max\{F(x_1 + t), 1 - F(r_{\frac{1}{2}} + t)\} \geq \max\{p_L, F(r_{\frac{1}{2}} + t) - F(x_1 + t) + q\}$ or (2ii) $\max\{F(x_1 + t), 1 - F(r_{\frac{1}{2}} + t)\} \geq \max\{p_H, F(r_{\frac{1}{2}} + t) - F(x_1 + t)\}$. Let the established party that locates at $x_1 < r_{\frac{1}{2}}$ under the strategy profile instead locate at $x_1 + \epsilon$ for $\epsilon > 0$ satisfying: (1) $1 - F(r_{\frac{1}{2}} + .5\epsilon) \geq p_H$, (2) $F(x_1 + \epsilon) < 1 - F(r_{\frac{1}{2}} + .5\epsilon)$, $F(r_{\frac{1}{2}} + .5\epsilon) > \max\{p_H, 1 - F(2r_{\frac{1}{2}} - x_1)\}$ and (3) for any $t \in (0, r_{\frac{1}{2}} - (x_1 - .5\epsilon))$, we have either (3i) $F(r_{\frac{1}{2}} + .5\epsilon + t) - F(x_1 + \epsilon + t) < \min\{F(x_1 + \epsilon + t), 1 - F(r_{\frac{1}{2}} + .5\epsilon + t)\}$, or (3ii) $F(r_{\frac{1}{2}} + .5\epsilon + t) - F(x_1 + \epsilon + t) \geq \min\{F(x_1 + \epsilon + t), 1 - F(r_{\frac{1}{2}} + .5\epsilon + t)\}$ and either (3iia) $\max\{F(x_1 + \epsilon + t), 1 - F(r_{\frac{1}{2}} + t + .5\epsilon)\} \geq \max\{p_L, F(r_{\frac{1}{2}} + .5\epsilon + t) - F(x_1 + \epsilon + t) + q\}$ or (3iib) $\max\{F(x_1 + \epsilon + t), 1 - F(r_{\frac{1}{2}} + .5\epsilon + t)\} \geq \max\{p_H, F(r_{\frac{1}{2}} + .5\epsilon + t) - F(x_1 + \epsilon + t)\}$. After this deviation, the challenger strictly prefers the action *out*, and the deviating party wins the election in the first round. □ We conclude that under a moving threshold, in every equilibrium profile $\mathcal{X}$ satisfying $\sum_{x \in X} \varrho(x) \leq 2$, we must have $a_c \in \mathbb{R}$. □